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X. *The Distribution of Stress Round a Circular Hole in a Plate.*

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(*Communicated by Prof.* E. G. Coker, *F.R.S.*)

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(1) *Introduction.*

The study of stress distributions in elastic plates would seem to have many important applications in engineering practice, and from this point of view it is, at first sight, surprising that our knowledge of the subject is not more detailed than it is at present. True, the fact that the stresses are derivable from a stress-function, and the equation satisfied by this stress-function, have long been known* ; particular solutions, satisfying the types of boundary condition met with in practice are, however, rare. Jeffery, in his paper, "Plane Stress and Plane Strain in Bipolar Co-ordinates,"† says that in the problem of the equilibrium of an elastic solid "knowledge comes by patient accumulation of special solutions rather than by the establishment of great general propositions"; and later, that "it is of considerable importance that the two-dimensional problems should be worked out more thoroughly." The present paper is an attempt to fill one gap by a fairly full examination of the stresses round a circular hole in an otherwise infinite elastic plate of uniform thickness, due to prescribed tractions in the plane of the plate, acting on the circular boundary. A general solution is obtained and particular cases are examined in detail, these cases being chosen to combine, as far as possible, mathematical simplicity with some semblance of the type of distribution of traction likely to occur in practice; the analysis is also applied to examine some experimental results obtained in the Engineering Laboratories of University College by Prof. E. G. Coker and T. Fukuda.

The attention of the author was first turned to this type of problem in 1919 by Prof. Coker, whose experimental method of solution is now well known. He suggested an attempt to calculate mathematically the stresses in the neighbourhood of a circular hole in a tension member. An exact solution was not obtained, and an approximate one is only applicable when the diameter of the hole is very small compared to the width of the member. In the course of the investigation, however, it became necessary to find the stresses due to a simple distribution of traction on the circular boundary, and difficulties were met with when the traction did not form a system in equilibrium.

* G. B. Airy, 'Brit. Assoc. Rep.,' 1862.

† 'Phil. Trans.,' A, vol. 221, p. 265 (1920).

 [Published July 2, 1928.

The clearing up of these difficulties led to the results which follow, the work having been carried out at intervals, between several lengthy interruptions, since.

The stresses in a circular ring have been studied in some detail by FILON,* and some of the present formulæ obtained are limiting cases of his. As, however, he considered at length only rings of width comparable with the radius, with a view to testing the validity of certain approximate methods used by engineers, it is thought better to give here the derivation of the formulæ *ab initio* as obtained independently by the present author.

(2) *Fundamental Equations.*

Referred to polar co-ordinates, r, θ, the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{\theta\theta}$, satisfy the two equations† :—

$$\left.\begin{aligned} \frac{d\,(\widehat{rr})}{dr} + \frac{1}{r}\frac{d\,(\widehat{r\theta})}{d\theta} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} &= 0 \\ \frac{d\,(\widehat{r\theta})}{dr} + \frac{1}{r}\frac{d\,(\widehat{\theta\theta})}{d\theta} + \frac{2\,\widehat{r\theta}}{r} &= 0 \end{aligned}\right\} \quad \ldots \ldots \quad (2.1)$$

It is known that the stresses can be expressed in terms of a stress-function, χ, by means of the equations—

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{r}\frac{d\chi}{dr} + \frac{1}{r^2}\frac{d^2\chi}{d\theta^2} \\ \widehat{\theta\theta} &= \frac{d^2\chi}{dr^2} \\ \widehat{r\theta} &= -\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{d\theta}\right) \end{aligned}\right\}, \quad \ldots \ldots \quad (2.2)$$

provided

$$\nabla^4\chi = 0. \quad \ldots \ldots \quad (2.3)$$

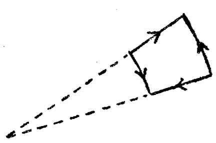

Fig. 1.

The direct stresses, $\widehat{rr}$ and $\widehat{\theta\theta}$, will be taken positive when they are tensions, and the shear stress, $\widehat{r\theta}$, acts when positive as indicated in fig. 1. The formulæ for the corresponding displacements are, in rectangular co-ordinates,

$$\left.\begin{aligned} u &= \frac{1}{E}\left\{\xi + \tfrac{1}{2}\sigma z^2\frac{d\Theta}{dx}\right\} - \frac{1+\sigma}{E}\frac{d\chi}{dx} \\ v &= \frac{1}{E}\left\{\eta + \tfrac{1}{2}\sigma z^2\frac{d\Theta}{dy}\right\} - \frac{1+\sigma}{E}\frac{d\chi}{dy} \\ w &= -\frac{\sigma}{E}z\Theta \end{aligned}\right\}, \quad \ldots \ldots \quad (2.4)$$

* 'Proc. Inst. Civ. Eng.,' No. 12 (1924).

† The equations are those of "generalised plane stress"; see Love's 'Theory of Elasticity,' p. 473 (3rd ed.).

x and y being co-ordinates in the plane of the plate; z the distance from the central surface; u, v, w, the displacements in the x, y, z, directions respectively; E, Young's Modulus; σ, Poisson's ratio;

$$\Theta = \widehat{rr} + \widehat{\theta\theta} = \nabla^2\chi,$$

and, if $f(\zeta)$ is a function of the complex variable $\zeta = x + \iota y$ whose real part is Θ,

$$\xi + \iota\eta = \int f(\zeta)\, d\zeta.$$

It has been shown by MICHELL* that the state of stress (and consequently χ) does not depend upon the elastic constants of the material of the plate, but only upon its shape and upon the applied tractions, *provided the force resultant of the tractions on any boundary is zero.* If, however, the resultant of the tractions *is* a force, then χ, and the stresses, involve POISSON's ratio. Further, when the region occupied by the plate is not simply connected, χ may be many-valued, but must, of course, give rise to a *single-valued* system of stresses and displacements. The effect of multiple connectivity has been considered in more detail by FILON,† who gave the general theory, and also the solution showing the effect of σ for the particular case of a circular ring. In dealing with a circular boundary, it is found that the exceptions just mentioned affect the terms of the first order only in the expansion in terms of cosines and sines of multiples of θ. We, therefore, first conduct a preliminary investigation of these terms.

(3) *Examination of the cyclic terms in the displacement formulæ.*

The most general value of χ, as regards such terms alone, satisfying equations (2.3), is :—

$$\chi = A_0 r\theta \sin\theta + B_0 r\theta \cos\theta + A_1 r \log r \cos\theta + B_1 r \log r \sin\theta \quad . \quad . \quad (3.1)$$

whence we obtain, by (2.2) and (2.4),

$$\left.\begin{aligned} \widehat{rr} &= (2A_0 \cos\theta - 2B_0 \sin\theta + A_1 \cos\theta + B_1 \sin\theta)/r \\ \widehat{\theta\theta} &= (A_1 \cos\theta + B_1 \sin\theta)/r \\ \widehat{r\theta} &= (A_1 \sin\theta - B_1 \cos\theta)/r \end{aligned}\right\} \quad . \quad . \quad . \quad (3.2)$$

$$\Theta = 2\{(A_0 + A_1)\cos\theta - (B_0 - B_1)\sin\theta\}/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

$$\left.\begin{aligned} \xi &= 2(A_0 + A_1)\log r + 2(B_0 - B_1)\,\theta \\ \eta &= 2(A_0 + A_1)\,\theta - 2(B_0 - B_1)\log r \end{aligned}\right\} \quad . \quad . \quad . \quad (3.4)$$

$$\left.\begin{aligned} u &= \{2(B_0 - B_1)\,\theta - (1+\sigma)\,B_0\theta + \ldots\}/E \\ v &= \{2(A_0 + A_1)\,\theta - (1+\sigma)\,A_0\theta + \ldots\}/E \end{aligned}\right\} \quad . \quad . \quad . \quad (3.5)$$

* 'Proc. Lond. Math. Soc.,' vol. 31, p. 100 (1900).

† 'Brit. Assoc. Rep.,' 1921.

where only the cyclic terms in the displacement formulæ have been written down. To ensure that the displacement is single-valued, we must have :—

$$B_1/B_0 = \tfrac{1}{2}(1-\sigma)\,;\ \ A_1/A_0 = -\tfrac{1}{2}(1-\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

The formulæ for the displacements have now served our purpose, and will not be further dealt with. A simple calculation, or indeed the known facts, will show that in any practical case, the displacements are very small.

(4) *Normal pressure varying as cos θ over the internal circular boundary of an otherwise infinite plate.*

As a first example, before embarking upon the general solution, we solve the problem for a normal pressure varying as cos θ applied to the boundary. This is of more practical importance than the fact that the traction is of opposite sign upon the two halves of the boundary would, at first sight, indicate. In conjunction with the known and very simple distribution of stress due to a uniform pressure on the boundary, it will give the stress distribution round a hole into which a rivet has been driven, when the rivet is pulled sideways (*i.e.*, by a force in the plane of the plate). Moreover, these terms will be the important ones at a distance from the hole, in *any* case. Finally, this is the simplest example of the exceptional cases mentioned in § (2), for the resultant of the tractions is evidently a force directed along the initial line, and the region occupied by the plate is doubly connected.

Denoting the radius of the hole by a, the stress function must be given by

$$\chi = A_0\{r\theta\sin\theta - \tfrac{1}{2}(1-\sigma)\,r\log r\cos\theta\} + Ca^2\cos\theta/r, \quad . \quad . \quad . \quad . \quad (4.1)$$

for this is evidently a solution of equation (2.3), even in θ, giving stresses which vanish at infinity ; the results of § (3) are incorporated, so that the displacements will be single valued ; and, when we notice that a term in $r\cos\theta$ is not necessary, as affecting neither the stresses, nor, essentially, the displacements, it is seen to be the most general solution of the requisite type. Using equations (2.2), we find for the stresses :—

$$\begin{aligned} \widehat{rr} &= \cos\theta\,\{\tfrac{1}{2}(3+\sigma)\,A_0 - 2Ca^2/r^2\}/r, \\ \widehat{\theta\theta} &= -\cos\theta\,\{\tfrac{1}{2}(1-\sigma)\,A_0 - 2Ca^2/r^2\}/r \\ \widehat{r\theta} &= -\sin\theta\,\{\tfrac{1}{2}(1-\sigma)\,A_0 + 2C\,a^2/r^2\}/r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2) \end{aligned}$$

If, at $r = a$, we have $\widehat{rr} = -P\cos\theta$, $\widehat{r\theta} = 0$, then—

$$\left.\begin{aligned} \tfrac{1}{2}(3+\sigma)\,A_0 - 2C &= -Pa \\ \tfrac{1}{2}(1-\sigma)\,A_0 + 2C &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

whence

$$A_0 = -\tfrac{1}{2}Pa, \qquad C = \tfrac{1}{8}(1-\sigma)\,Pa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.31)$$

giving

$$\chi = \tfrac{1}{2} \mathrm{P}a \{\tfrac{1}{2}(1-\sigma) r \log r \cos\theta - r\theta \sin\theta + \tfrac{1}{4}(1-\sigma) a^2 \cos\theta / r\}, \quad (4.4)$$

$$\left.\begin{aligned} \widehat{rr} &= -\mathrm{P}\frac{a}{r}\cos\theta\left\{1 - \frac{1-\sigma}{4}\left(1-\frac{a^2}{r^2}\right)\right\} \\ \widehat{\theta\theta} &= \mathrm{P}\frac{a}{r}\cos\theta\,\frac{1-\sigma}{4}\left(1+\frac{a^2}{r^2}\right) \\ \widehat{r\theta} &= \mathrm{P}\frac{a}{r}\sin\theta\,\frac{1-\sigma}{4}\left(1-\frac{a^2}{r^2}\right) \end{aligned}\right\} \quad (4.5)$$

The resultant force, X, is given by—

$$\mathrm{X} = \int_0^{2\pi} \mathrm{P}\cos^2\theta \, . \, a d\theta = \pi \mathrm{P}a. \quad (4.6)$$

So that, in terms of X, the stress-function and the stresses are given by :—

$$\chi = \frac{\mathrm{X}}{2\pi}\left\{\tfrac{1}{2}(1-\sigma) r \log r \cos\theta - r\theta\sin\theta + \tfrac{1}{4}(1-\sigma)\frac{a^2}{r}\cos\theta\right\}, \quad (4.41)$$

$$\left.\begin{aligned} \widehat{rr} &= -\frac{\mathrm{X}}{\pi}\frac{\cos\theta}{r}\left\{1 - \frac{1-\sigma}{4}\left(1-\frac{a^2}{r^2}\right)\right\} \\ \widehat{\theta\theta} &= \frac{\mathrm{X}(1-\sigma)}{4\pi}\frac{\cos\theta}{r}\left(1+\frac{a^2}{r^2}\right) \\ \widehat{r\theta} &= \frac{\mathrm{X}(1-\sigma)}{4\pi}\frac{\sin\theta}{r}\left(1-\frac{a^2}{r^2}\right) \end{aligned}\right\} \quad (4.51)$$

From these we see that in going from the experimental results on a substance for which POISSON's ratio is σ_0 to another for which the ratio is σ_1, we must apply the corrections

$$\widehat{rr}_1 - \widehat{rr}_0 = \frac{(\sigma_1-\sigma_0)\mathrm{X}}{4\pi r}\left(1-\frac{a^2}{r^2}\right)\cos\theta,$$

$$\widehat{\theta\theta}_1 - \widehat{\theta\theta}_0 = -\frac{(\sigma_1-\sigma_0)\mathrm{X}}{4\pi r}\left(1+\frac{a^2}{r^2}\right)\cos\theta,$$

$$\widehat{r\theta}_1 - \widehat{r\theta}_0 = -\frac{(\sigma_1-\sigma_0)\mathrm{X}}{4\pi r}\left(1-\frac{a^2}{r^2}\right)\sin\theta,$$

which agree with those of FILON* if the outer radius of the ring is made infinite.

The variation of these stresses with r/a is shown by the values tabulated in Table I, and the corresponding graphs, fig. 2. Two distinct values of σ have been used, namely,

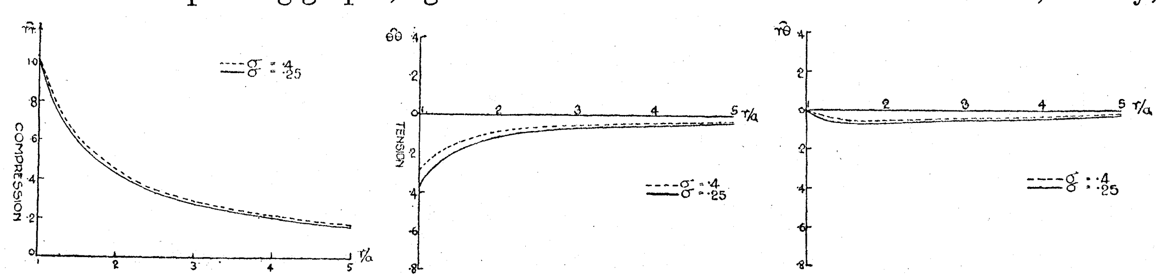

Fig. 2.

* 'Brit. Assoc. Rep.,' 1921.

0·25 (approximately that for steel) and 0·4 (approximately that for nitro-cellulose as used by COKER in his "Photo-Elastic measurements").* In order that the numerical values to be given for various distributions of boundary traction may be easily comparable, it is desirable to give the values of the stresses in terms of a unit proportional to the most important quantity common to the series, viz., the force-resultant X, of the system of tractions. Convenience of calculation pointed to the choice of $X/\pi a$ as the unit, so the stresses are obtained by the multiplication of $X/\pi a$ by the number tabulated.

TABLE I.

$$\widehat{rr}_e = -\text{P} \cos \theta, \quad \widehat{r\theta}_e = 0.$$

r/a.	$\widehat{rr}/\cos\theta$.		$\widehat{\theta\theta}/\cos\theta$.		$\widehat{r\theta}/\sin\theta$.	
	$\sigma = 0{\cdot}25$.	$\sigma = 0{\cdot}4$.	$\sigma = 0{\cdot}25$	$\sigma = 0{\cdot}4$.	$\sigma = 0{\cdot}25$.	$\sigma = 0{\cdot}4$.
1·0	−1·000	−1·000	+0·375	+0·300	+0·000	+0·000
1·1	0·880	0·885	0·311	0·249	0·030	0·024
1·2	0·786	0·795	0·265	0·212	0·047	0·038
1·5	0·597	0·611	0·181	0·144	0·069	0·056
2·0	0·430	0·444	0·117	0·094	0·070	0·056
2·5	0·337	0·350	0·084	0·069	0·063	0·050
3·0	0·278	0·289	0·069	0·055	0·056	0·044
4·0	0·206	0·215	0·050	0·040	0·043	0·036
5·0	0·164	0·171	0·039	0·031	0·036	0·029

These figures and graphs bring out one very important point—important, that is, from the practical standpoint—namely, the comparative insignificance of the effects of the variations of σ likely to be met with. The greatest difference between the values of $\widehat{rr}$ or $\widehat{r\theta}$ at any point due to the variation of σ from 0·25 to 0·4 is only 1·4 per cent. of the maximum stress—the difference in the case of $\widehat{\theta\theta}$ is greater, just over 7 per cent., but still is of little *practical* importance. When we remember that, with other distributions of boundary traction, there will be other terms in the stress formulæ *which will not contain* σ, and so the effects of σ may confidently be expected to be, in proportion, less still, it is seen that the effects of POISSON'S ratio are, from a practical standpoint, negligible. Consequently, experimental results obtained by measuring the stresses in nitro-cellulose or glass models can be accepted as giving valid results for similar members or structures of steel, since the effect of the difference of POISSON'S ratio is probably less than the order of the possible experimental error. The general effect of increasing σ is seen to be in the direction of easing the stresses.

In the same way, we may obtain the solution for the case of tangential boundary

* 'Trans. Inst. Eng. Shipbuilders Scot.,' 1920.

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traction varying as sin θ, but we anticipate by picking out the terms from the general solution to be obtained later (see § (5)). Taking $\widehat{r\theta}_e = \mathrm{P} \sin\theta$, we find :—

$$\widehat{rr} = -\tfrac{1}{4}(3+\sigma)\,\mathrm{P}a\,(1 - a^2/r^2)\cos\theta/r$$

$$\widehat{\theta\theta} = \mathrm{P}\{1 - \tfrac{1}{4}(3+\sigma)(1 + a^2/r^2)\}\,a\cos\theta/r$$

$$\widehat{r\theta} = \mathrm{P}\{1 - \tfrac{1}{4}(3+\sigma)(1 - a^2/r^2)\}\,a\sin\theta/r.$$

The resultant of the system of tractions is a force, of amount $\pi \mathrm{P}a$, acting along the line $\theta = 0$. Numerical values are given in Table II, and the corresponding graphs in fig. 3.

Table II.

$\widehat{rr}_e = 0, \qquad \widehat{r\theta}_e = \mathrm{P}\sin\theta.$

r/a.	$\widehat{rr}/\cos\theta$.		$\widehat{\theta\theta}/\cos\theta$.		$\widehat{r\theta}/\sin\theta$.	
	$\sigma = 0{\cdot}25$.	$\sigma = 0{\cdot}4$.	$\sigma = 0{\cdot}25$.	$\sigma = 0{\cdot}4$.	$\sigma = 0{\cdot}25$.	$\sigma = 0{\cdot}4$.
1·0	−0·000	−0·000	−0·625	−0·700	+1·000	+1·000
1·1	0·128	0·134	0·440	0·502	0·781	0·775
1·2	0·207	0·216	0·314	0·367	0·626	0·617
1·5	0·301	0·316	0·116	0·152	0·366	0·352
2·0	0·305	0·319	0·006	0·031	0·195	0·161
2·5	0·273	0·286	+0·023	+0·006	0·127	0·114
3·0	0·241	0·252	0·032	0·018	0·093	0·081
4·0	0·190	0·199	0·034	0·024	0·060	0·051
5·0	0·156	0·163	0·031	0·023	0·044	0·027

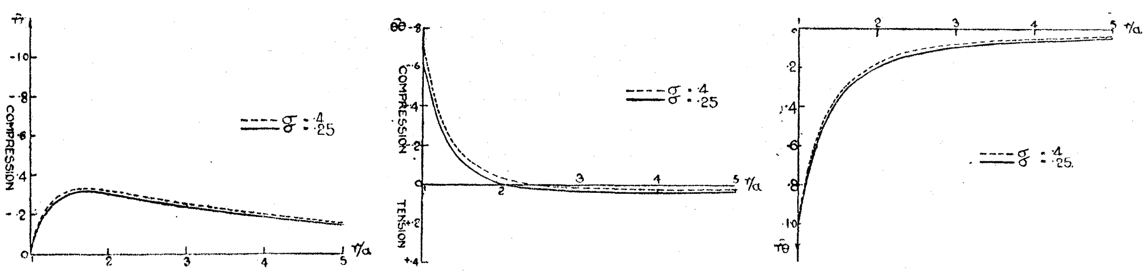

Fig. 3.

Again, the effects of σ are very small indeed; and so, in all that follows, we shall use as the value of σ, 0·25.

It is now possible to give some idea of the stress distribution round a rivet driven into a hole. As an instance, take the edge stresses to be given by

$$\widehat{rr}_e = -\mathrm{P}(1 + \cos\theta), \quad \widehat{r\theta}_e = \tfrac{1}{4}\mathrm{P}\sin\theta.$$

The constant term in $\widehat{rr}_e$ gives :—

$$\widehat{rr} = -\mathrm{P}a^2/r^2, \quad \widehat{\theta\theta} = \mathrm{P}a^2/r^2, \quad \widehat{r\theta} = 0,$$

and the stresses due to the other terms can be obtained from Tables I and II. The stresses are given in Table III and the corresponding graphs in fig. 4.

TABLE III.

$$\widehat{rr}_e = -\mathrm{P}(1 + \cos\theta), \quad \widehat{r\theta}_e = \tfrac{1}{4}\mathrm{P}\sin\theta.$$

θ	0°	30°	60°	90°	120°	150°	180°
r/a				$\widehat{rr}$			
1·0	−1·600	−1·493	−1·200	−0·800	−0·400	−0·107	−0·000
1·1	1·391	1·301	1·026	0·661	0·296	0·021	0·069
1·2	1·226	1·136	0·891	0·556	0·221	+0·024	0·114
1·5	0·894	0·822	0·625	0·356	0·087	0·110	0·182
2·0	0·605	0·555	0·402	0·200	+0·002	0·155	0·205
2·5	0·452	0·409	0·290	0·128	0·034	0·153	0·196
3·0	0·359	0·323	0·224	0·089	0·046	0·145	0·181
4·0	0·252	0·225	0·151	0·050	0·051	0·125	0·152
5·0	0·194	0·172	0·113	0·032	0·049	0·108	0·130
				$\widehat{\theta\theta}$			
1·0	+0·624	+0·648	+0·712	+0·800	+0·888	+0·952	+0·976
1·1	0·500	0·522	0·580	0·661	0·742	0·800	0·822
1·2	0·404	0·425	0·480	0·556	0·632	0·687	0·708
1·5	0·234	0·250	0·295	0·356	0·417	0·462	0·478
2·0	0·108	0·120	0·154	0·200	0·246	0·280	0·292
2·5	0·056	0·066	0·092	0·128	0·164	0·190	0·200
3·0	0·027	0·035	0·058	0·089	0·120	0·143	0·151
4·0	0·004	0·010	0·027	0·050	0·073	0·090	0·096
5·0	−0·026	−0·001	0·013	0·032	0·051	0·065	0·070
				$\widehat{r\theta}$			
1·0				0·200			
1·1				0·220			
1·2				0·203			
1·5				0·160			
2·0		× sin θ		0·119		× sin θ	
2·5				0·095			
3·0				0·079			
4·0				0·058			
5·0				0·047			

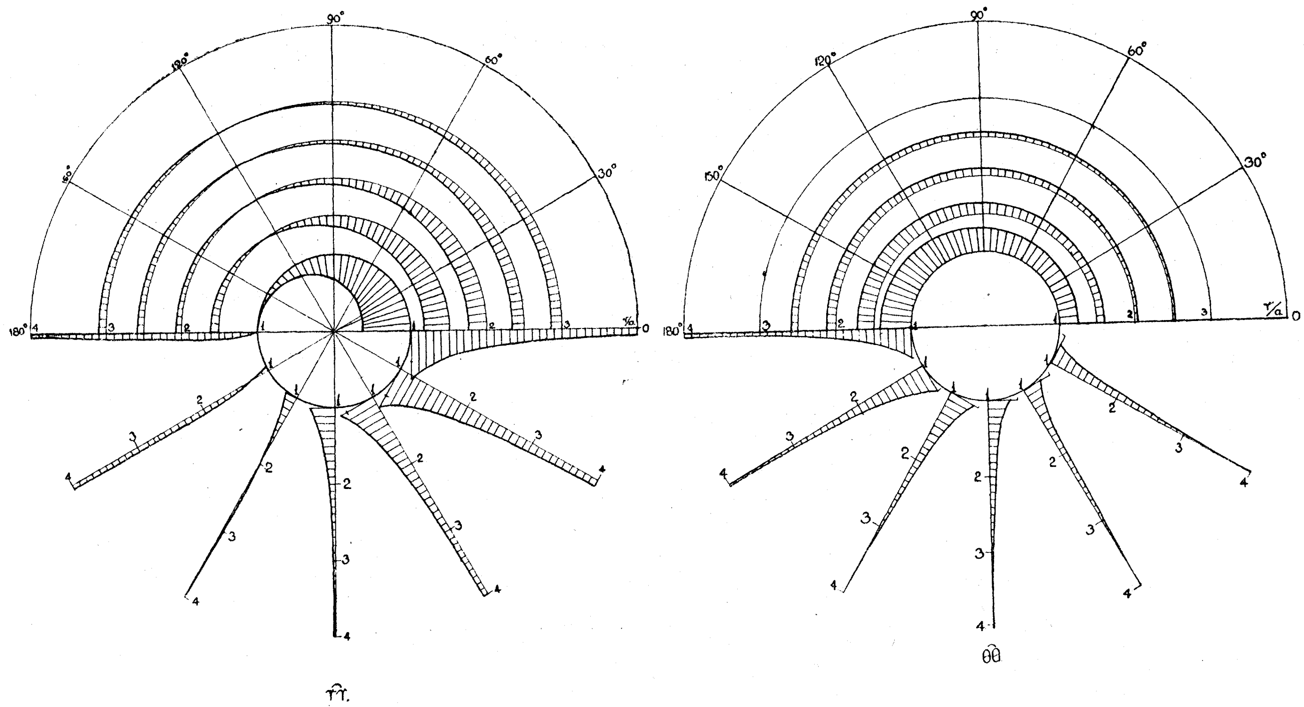

Fig. 4.

(5) *Stress distribution in an infinite plate due to forces in the plane of the plate applied to an internal circular boundary.*

Having determined the form of the cyclic terms in χ, to avoid cyclic terms in the displacement formulæ, and worked out results in some very simple cases, we now come to the main problem, the distribution of stress in an infinite, uniformly thick, elastic plate, containing a circular hole, due to tractions applied in the plane of the plate, to the boundary of the hole; the plate being free from stress at infinity.

We take, as the most general solution of (2.3) appropriate to the case in view, and incorporating the results of § (3) :—

$$\chi = a^2 \Big[A_0 \log (r/a) + B_0\theta + A_1 \{r/a \,.\, \theta \sin\theta - \tfrac{1}{2}(1-\sigma)\, r/a \,.\, \log (r/a) \cos\theta\} \\ + B_1 \{r/a \,.\, \theta\cos\theta + \tfrac{1}{2}(1-\sigma)\, r/a \,.\, \log(r/a) \sin\theta\} + C_1 a\cos\theta + D_1 a \sin\theta/r \\ + \sum_2^\infty \{a^{m-2}\, r^{-m+2} (A_m \cos m\theta + B_m \sin m\theta) \\ + a^m r^{-m} (C_m \cos m\theta + D_m \sin m\theta)\}\Big] \quad \ldots \quad (5.1)$$

giving :—

$$\widehat{rr} = A_0 a^2/r^2 + \tfrac{1}{2}(3+\sigma)\, a/r \,.\, (A_1\cos\theta - B_1 \sin\theta) - 2a^3/r^3 \,.\, (C_1\cos\theta + D_1\sin\theta) \\ - \Sigma\,\{(m+2)(m-1)\, a^m\, r^{-m} (A_m\cos m\theta + B_m \sin m\theta) \\ + m(m+1)\, a^{m+2}\, r^{-m-2} (C_m \cos m\theta + D_m \sin m\theta)\} \quad \ldots \quad (5.21)$$

$$\widehat{\theta\theta} = -A_0 a^2/r^2 - \tfrac{1}{2}(1-\sigma)\, a/r \,.\, (A_1\cos\theta - B_1\sin\theta) + 2a^3/r^3 \,.\, (C_1\cos\theta + D_1\sin\theta) \\ + \Sigma\,\{(m-2)(m-1)\, a^m r^{-m} (A_m\cos m\theta + B_m\sin m\theta) \\ + m(m+1)\, a^{m+2}\, r^{-m-2} (C_m\cos m\,\theta + D_m \sin m\theta)\} \quad \ldots \quad (5.22)$$

$$\widehat{r\theta} = -B_0 a^2/r^2 - \tfrac{1}{2}(1-\sigma) a/r \,.\, (A_1 \sin\theta + B_1 \cos\theta) - 2a^3/r^3 \,.\, (C_1 \sin\theta - D_1 \cos\theta)$$
$$+ \Sigma \{ m(m-1)\, a^m r^{-m} (-A_m \sin m\theta + B_m \cos m\theta)$$
$$+ m(m+1)\, a^{m+2} r^{-m-2} (-C_m \sin m\theta + D_m \cos m\theta)\} . \quad . \quad . \quad . \quad . \quad . \quad (5.23)$$

The coefficients A_m, B_m, C_m, D_m, must be determined so that the values of the stresses $\widehat{rr}$ and $\widehat{r\theta}$ at the boundary of the hole agree with those prescribed, and it will be found that there are just sufficient of them for this to be done. We take, as the most general formulæ for the edge tractions :—

$$\left. \begin{aligned} -\widehat{rr}_e &= R_0 + \sum_{m=1}^{\infty} (R_m \cos m\theta + S_m \sin m\theta) \\ -\widehat{r\theta}_e &= T_0 + \sum_{m=1}^{\infty} (T_m \cos m\theta + U_m \sin m\theta) \end{aligned} \right\} . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

Putting $r = a$ in equations (5.21) and (5.23) and equating coefficients of $\cos m\theta$ and $\sin m\theta$ in these and (5.3), we obtain the following set of equations to determine the co-efficients of (5.1) :—

$$\left. \begin{aligned} R_0 &= -A_0 & T_0 &= B_0 \\ R_1 &= -\tfrac{1}{2}(3+\sigma) A_1 + 2C_1 & T_1 &= \tfrac{1}{2}(1-\sigma) B_1 - 2D_1 \\ S_1 &= \tfrac{1}{2}(3+\sigma) B_1 + 2D_1 & U_1 &= \tfrac{1}{2}(1-\sigma) A_1 + 2C_1 \\ R_m &= (m+2)(m-1) A_m + m(m+1) C_m && \\ S_m &= (m+2)(m-1) B_m + m(m+1) D_m && \\ T_m &= -m(m-1) B_m - m(m+1) D_m && \\ U_m &= m(m-1) A_m + m(m+1) C_m && \end{aligned} \right\}, \quad (5.41)$$

and, consequently,

$$\left. \begin{aligned} A_0 &= -R_0 & B_0 &= T_0 \\ A_1 &= -\tfrac{1}{2}(R_1 - U_1) & C_1 &= \tfrac{1}{8}\{(1-\sigma) R_1 + (3+\sigma) U_1\} \\ B_1 &= \tfrac{1}{2}(S_1 + T_1) & D_1 &= \tfrac{1}{8}\{(1-\sigma) S_1 - (3+\sigma) T_1\} \\ A_m &= (R_m - U_m)/2(m-1) & C_m &= \{-mR_m + (m+2) U_m\}/2m(m+1) \\ B_m &= (S_m + T_m)/2(m-1) & D_m &= -\{mS_m + (m+2) T_m\}/2m(m+1) \end{aligned} \right\} . \quad (5.42)$$

Using these values of the coefficients, we obtain the following formulæ for the stresses :—

$$\widehat{rr} = -\sum_{0}^{\infty} \tfrac{1}{2} a^m r^{-m} \cos m\theta \,\{2R_m - 2U_m (1 - a^2/r^2) + m(R_m - U_m)(1 - a^2/r^2)\}$$
$$+ R_0 (1 - a^2/r^2) + \tfrac{1}{4}(3-\sigma)(R_1 - U_1)(1 - a^2/r^2)\, a \cos\theta/r$$
$$- \Sigma\, \tfrac{1}{2} a^m r^{-m} \sin m\theta \,\{2S_m + 2T_m (1 - a^2/r^2) + m(S_m + T_m)(1 - a^2/r^2)\}$$
$$+ \tfrac{1}{4}(3-\sigma)(S_1 + T_1)(1 - a^2/r^2)\, a \sin\theta/r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.51)$$

$$\widehat{\theta\theta} = \sum_0^\infty \tfrac{1}{2}a^m r^{-m} \cos m\theta \{m(R_m - U_m)(1 - a^2/r^2) - 2R_m + 2U_m(1 + a^2/r^2)\}$$
$$+ R_0(1 + a^2/r^2) + \tfrac{1}{4}(3 - \sigma)(R_1 - U_1)(1 + a^2/r^2)\, a \cos\theta/r$$
$$+ \sum_0^\infty \tfrac{1}{2}a^m r^{-m} \sin m\theta \{m(S_m + T_m)(1 - a^2/r^2) - 2S_m - 2T_m(1 + a^2/r^2)\}$$
$$+ \tfrac{1}{4}(3 - \sigma)(S_1 + T_1)(1 + a^2/r^2) \sin\theta, \quad \ldots \quad (5.52)$$

$$\widehat{r\theta} = - \Sigma\, \tfrac{1}{2}a^m r^{-m} \sin m\theta \{m(R_m - U_m)(1 - a^2/r^2) + 2U_m\, a^2/r^2\}$$
$$+ \tfrac{1}{4}(3 - \sigma)(R_1 - U_1)(1 - a^2/r^2)\, a \sin\theta/r$$
$$+ \Sigma\, \tfrac{1}{2}a^m r^{-m} \cos m\theta \{m(S_m + T_m)(1 - a^2/r^2) - 2T_m\, a^2/r^2\}$$
$$- \tfrac{1}{4}(3 - \sigma)(S_1 + T_1)(1 - a^2/r^2)\, a \cos\theta/r. \quad \ldots \quad (5.53)$$

The formula for the sum of the principal stresses is also of use ; it is :—

$$\widehat{rr} + \widehat{\theta\theta} = 2R_0 + \tfrac{1}{2}(3 - \sigma)(R_1 - U_1)\, a \cos\theta/r + \tfrac{1}{2}(3 - \sigma)(S_1 + T_1)\, a \sin\theta/r$$
$$- 2\,\Sigma\,(R_m - U_m)\, a^m r^{-m} \cos m\theta - 2\,\Sigma\,(S_m + T_m)\, a^m r^{-m} \sin m\theta. \quad \ldots \quad (5.54)$$

This is the complete solution, giving the stresses in terms of the boundary tractions, for the most general distribution of this traction. It is at present somewhat unwieldly in form, but will be made more concise later. It is evident that the terms in the summations are connections with the FOURIER series, for $\widehat{rr}_e$ and $\widehat{r\theta}_e$, and we shall find these can be expressed in terms of two pairs of conjugate functions. The terms outside the summations, in which alone σ occurs, are evidently due to the unbalanced part of the distribution, and so can be expressed in terms of the " singularities," *i.e.*, the average pressure on the boundary, the force resultant, and the couple resultant of the system. (As a matter of fact, the couple will not appear explicitly in the final formulæ.)

(6) *Singularities.*

As mentioned in the preceding paragraph, these are the average pressure on the boundary, P ; the components of the resultant force, X and Y ; the moment of the tractions about the centre of the hole, M. They are given by :—

$$P = \frac{1}{2\pi}\int_0^{2\pi} (-\widehat{rr}_e)\, d\theta = R_0. \quad \ldots \quad (6.1)$$

$$X = \int_0^{2\pi} \{-\widehat{rr}_e \cos\theta + \widehat{r\theta}_e \sin\theta\}\, a\,.\, d\theta$$
$$= \pi a\,(R_1 - U_1).$$

Therefore

$$R_1 - U_1 = X/\pi a, \quad \ldots \quad (6.21)$$

and, similarly,

$$S_1 + T_1 = Y/\pi a, \quad \ldots \quad (6.22)$$

$$M = \int_0^{2\pi} (-\widehat{r\theta}_e)\, a^2\,.\, d\theta = 2\pi a^2 T_0,$$

$$T_0 = M/2\pi a^2. \quad \ldots \quad (6.3)$$

(7) *Expression of the solution in terms of plane harmonic functions.*

An examination of the terms in the summations in equations (5.51), (5.52) and (5.53) will show that it is possible to express these sums in terms of plane harmonic functions which reduce to $\widehat{rr}_e$ and $\widehat{r\theta}_e$ at the edge of the hole, and the functions conjugate to these. To do this, introduce two functions of the complex variable, $z = re^{\iota\theta}/a$ defined by :—

$$V = \phi + \iota\phi' = \sum_0^\infty (R_m + \iota S_m)\, a^m r^{-m} e^{-\iota m\theta}, \qquad (7.11)$$

$$W = \psi + \iota\psi' = \sum_0^\infty (-U_m + \iota T_m)\, a^m r^{-m} e^{-\iota m\theta} \qquad (7.12)$$

in terms of which it is easy to verify that :—

$$\widehat{rr} = -(\phi + \psi) + \frac{a^2}{r^2}\psi + \tfrac{1}{2}\left(1 - \frac{a^2}{r^2}\right)\frac{\partial}{\partial\theta}(\phi' + \psi')$$
$$+ P\left(1 - \frac{a^2}{r^2}\right) + \frac{3-\sigma}{4\pi a}\cdot\frac{a}{r}\left(1 - \frac{a^2}{r^2}\right)(X\cos\theta + Y\sin\theta), \qquad (7.21)$$

$$\widehat{\theta\theta} = -(\phi + \psi) - \frac{a^2}{r^2}\psi - \tfrac{1}{2}\left(1 - \frac{a^2}{r^2}\right)\frac{\partial}{\partial\theta}(\phi' + \psi')$$
$$+ P\left(1 + \frac{a^2}{r^2}\right) + \frac{3-\sigma}{4\pi a}\cdot\frac{a}{r}\left(1 + \frac{a^2}{r^2}\right)(X\cos\theta + Y\sin\theta), \qquad (7.22)$$

$$\widehat{r\theta} = -\frac{a^2}{r^2}\psi' + \tfrac{1}{2}\left(1 - \frac{a^2}{r^2}\right)\frac{\partial}{\partial\theta}(\phi + \psi)$$
$$+ \frac{3-\sigma}{4\pi a}\cdot\frac{a}{r}\left(1 - \frac{a^2}{r^2}\right)(X\sin\theta - Y\cos\theta), \qquad (7.23)$$

It is evident, from their definitions in (7.11) and (7.12) that the four functions ϕ, ϕ', ψ and ψ' are completely determined by the system of applied tractions. It will be shown later that V and W are expressible as definite integrals, but in most of the examples chosen it has been simpler to determine the FOURIER co-efficients, and then sum the series (7.11) and (7.12). In fact, except in the simplest cases, and sometimes for values of r/a not much greater than 1, the stresses are calculated from the formulæ of paragraph (5) with less trouble than by those just given. Equally compact formulæ for χ and the displacements have not been found, but χ is only a mathematical convenience, with no obvious physical significance, and the displacements, as has been mentioned, are small and of little account in practice.

Remembering the relations, which hold for conjugate functions :—

$$\frac{\partial(\phi', \psi')}{\partial r} = -\frac{1}{r}\frac{\partial(\phi, \psi)}{\partial\theta}; \qquad \frac{\partial(\phi, \psi)}{\partial r} = \frac{1}{r}\frac{\partial(\phi', \psi')}{\partial\theta}, \qquad (7.3)$$

it is easily verified that the formulæ of this paragraph satisfy the fundamental equations of equilibrium, (2.1).

It is worth while giving here the formulæ for the sum of the principal stresses :—

$$\widehat{rr} + \widehat{\theta\theta} = -2(\phi + \psi) + 2\mathrm{P} + \frac{3-\sigma}{2\pi a} \cdot \frac{a}{r}(\mathrm{X}\cos\theta + \mathrm{Y}\sin\theta), \quad . \quad . \quad (7.4)$$

and for the hoop stress at the edge of the hole (to calculate which by the formula (5.52) is seldom easy as the series converges slowly when $r/a = 1$) :—

$$\widehat{\theta\theta}_e = -\phi - 2\psi + 2\mathrm{P} + (3-\sigma)/2\pi a \,.\, (\mathrm{X}\cos\theta + \mathrm{Y}\sin\theta).$$

(8) *Expression of the functions V and W as definite integrals.*

From their definition, V and W are functions of a complex variable

$$z = re^{\iota\theta}/a = x + \iota y,$$

finite at infinity, and such that the real part of V and the imaginary part of W are respectively equal to $-\widehat{rr}_e$ and $-\widehat{r\theta}_e$ at the edge of the hole. By a known theorem,* therefore :—

$$\mathrm{V} = \frac{1}{2\pi}\int_0^{2\pi} \frac{ze^{-\iota\gamma}+1}{ze^{-\iota\gamma}-1}(\widehat{rr}_e)\,d\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.1)$$

$$\mathrm{W} = \frac{\iota}{2\pi}\int_0^{2\pi} \frac{ze^{-\iota\gamma}+1}{ze^{-\iota\gamma}-1}(\widehat{r\theta}_e)\,d\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.2)$$

γ being the angular co-ordinate of the point of the boundary at which the values of $\widehat{rr}$ and $\widehat{r\theta}$ are taken. A physical interpretation of these integrals will be given later (see paragraph (9) below).

(9) *Single force, F, acting at* $(a, 0)$ *in the direction* $\theta = 0$.

The first example we choose is that of the stresses due to a radial force applied at one point of the boundary. Of course, infinities of stress at the point of application are to be expected, and the case can never be practically realised. There are, however, features which make it worthy of attention.

Choosing the radius through the point of application of the force as the initial line, we obtain for the Fourier coefficients :—

$$\mathrm{R}_0 = \mathrm{F}/2\pi a, \qquad \mathrm{R}_m = \mathrm{F}/\pi a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.1)$$

S. T. U. all zero, giving

$$\mathrm{V} = \frac{\mathrm{F}}{2\pi a}\,\frac{1 + a\,e^{-\iota\theta}/r}{1 - a\,e^{-\iota\theta}/r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.21)$$

$$\phi = \frac{\mathrm{F}}{2\pi a}\,\frac{1 - a^2/r^2}{1 - 2a\cos\theta/r + a^2/r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.22)$$

$$\phi' = -\frac{\mathrm{F}}{\pi a}\,\frac{a\sin\theta/r}{1 - 2a\cos\theta/r + a^2/r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.23)$$

* Forsyth, 'Theory of Functions' (2nd ed.), p. 440.

and for the "singularities,"

$$P = F/\pi a, \qquad X = F. \qquad \qquad (9.3)$$

The stresses can now be found by means of the formulæ (7.21)—(7.25). Values of the stresses (as multiples of $X/\pi a$) are given in Table IV, and the corresponding graphs in fig. 5 ($\sigma = 0{\cdot}25$).

TABLE IV.—*Single concentrated force.*

θ.	0°.	5°.	10°.	30°.	60°.	90°.	120°.	160°.	180°.
r/a.					$\widehat{rr}$.				
1·0									
1·2	−9·743	−7·189	−2·642	+0·098	+0·204	+0·128	+0·055	−0·007	−0·030
1·5	3·634	3·303	2·446	−0·279	0·259	0·204	0·104	+0·021	0·020
2·0	1·619			0·484	0·129	0·195	0·131	0·062	+0·034
2·5	0·982			0·436	0·047	0·158	0·131	0·081	0·064
3·0	0·685			0·355	0·002	0·124	0·122	0·086	0·074
4·0	0·412			0·261	−0·033	0·079	0·101	0·090	0·082
5·0	0·298			0·188	0·026	0·054	0·083	0·084	0·061
					$\widehat{\theta\theta}$.				
1·0		+2·370	+2·354	+2·191	+1·687	+1·000	+0·313	−0·191	−0·375
1·2	+0·901	−0·021	−0·983	0·677	1·015	0·675	0·251	0·083	0·207
1·5	0·551	+0·440	+0·219	0·159	0·525	0·412	0·174	0·033	0·107
2·0	0·305			0·125	0·215	0·205	0·096	0·011	0·055
2·5	0·199			0·115	0·123	0·118	0·056	0·011	0·039
3·0	0·144			0·097	0·074	0·071	0·044	0·015	0·032
4·0	0·089			0·069	0·051	0·028	0·013	0·015	0·023
5·0	0·063			0·042	0·041	0·023	0·004	0·015	0·023
					$\widehat{r\theta}$.				
1·2		−3·087	−2·366	−0·221	+0·106	+0·160	+0·146	+0·088	
1·5		0·642	0·991	0·486	0·081	0·205	0·200	0·120	
2·0				0·348	0·006	0·148	0·183	0·113	
2·5				0·207	−0·011	0·127	0·150	0·095	
3·0				0·129	0·012	0·097	0·122	0·079	
4·0				0·058	0·006	0·064	0·084	0·056	
5·0				0·030	+0·001	0·047	0·062	0·042	

The main interest of this case is, however, mathematical; it lies in the result of a comparison of the above formula for V with the integral formula (8.1). This supplies an immediate physical interpretation of the integral formula as a summation of the effects of "stress-sources" distributed along the boundary with line-density $\widehat{rr}_e$.

A similar interpretation can, of course, be given to (8.2).

(10) *Uniform radial pressure along an arc.*

As a next example we will consider the case of a uniform radial pressure distributed over an arc of the boundary. This will avoid the infinities of stress, and though it can

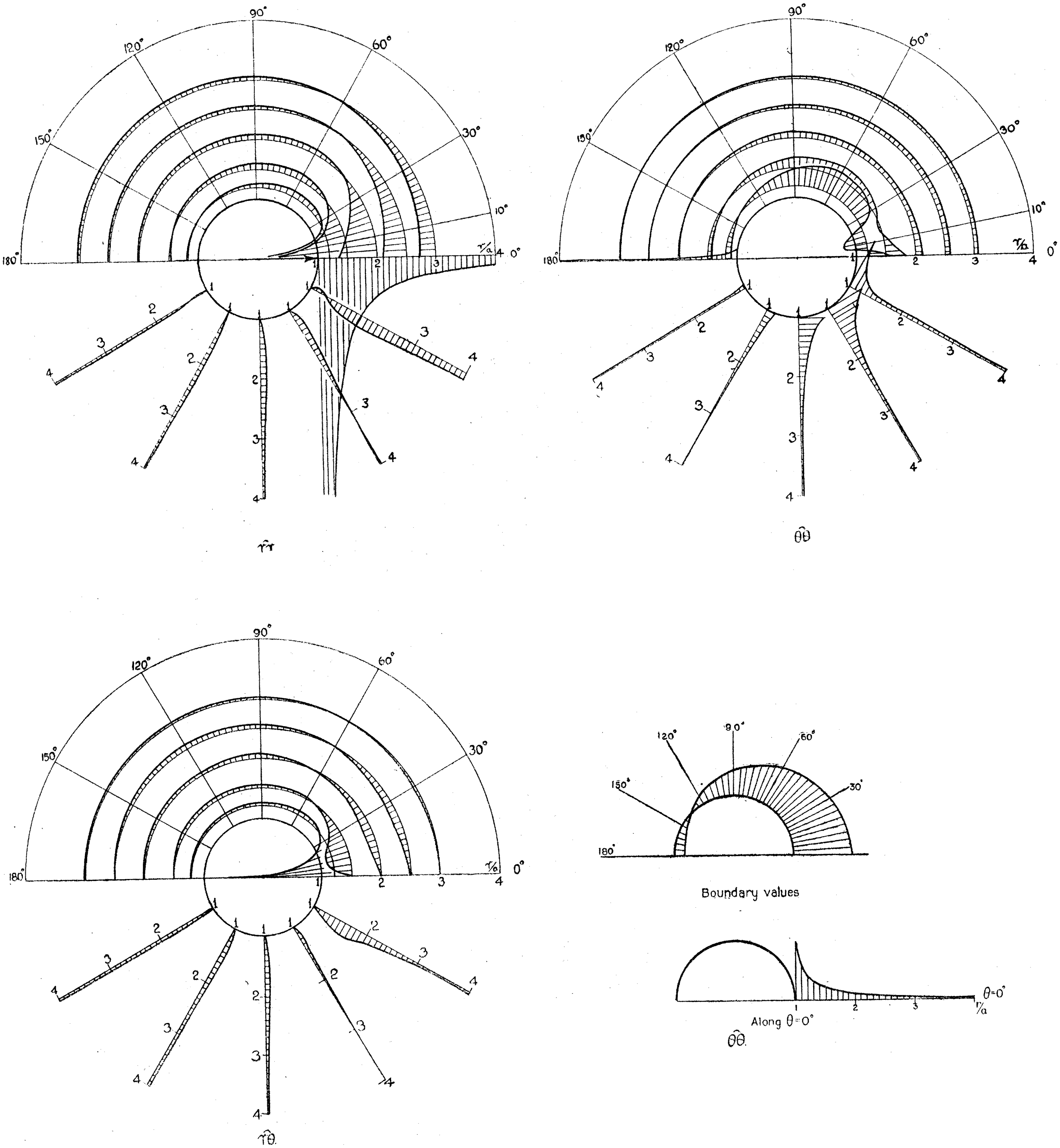

Fig. 5.

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only be approximately realised in practice, will serve to show the effects of a discontinuous distribution of boundary stress.

We take—

$$\widehat{rr}_e = -p \quad \text{if} \quad |\theta| \leq \alpha, \qquad \widehat{rr}_e = 0 \quad \text{if} \quad |\theta| > \alpha.$$

The FOURIER coefficients are

$$R_0 = \frac{p\alpha}{\pi}, \qquad R_m = \frac{2p}{\pi}\frac{\sin m\alpha}{m}, \quad \ldots \ldots \ldots \quad (10.1)$$

so that

$$V = \frac{p\alpha}{\pi} + \frac{2p}{\pi}\tan^{-1}\frac{a/r\,.\,e^{-\iota\theta}\sin\alpha}{1 - a/r\,.\,e^{-\iota\theta}\cos\alpha}, \quad \ldots \ldots \quad (10.2)$$

$$\phi = \frac{p}{\pi}\left\{\alpha + \tan^{-1}\frac{a/r\,.\,\sin(\theta+\alpha)}{1 - a/r\,.\,\cos(\theta+\alpha)} - \tan^{-1}\frac{a/r\,.\,\sin(\theta-\alpha)}{1 - a/r\,.\,\cos(\theta-\alpha)}\right\}, \quad (10.21)$$

$$\phi' = \frac{p}{2\pi}\log_e\frac{1 - 2a/r\,.\,\cos(\theta+\alpha) + a^2/r^2}{1 - 2a/r\,.\,\cos(\theta-\alpha) + a^2/r^2}, \quad \ldots \ldots \quad (10.22)$$

giving the stresses—

$$\widehat{rr} = -\frac{p\alpha}{\pi}\frac{a^2}{r^2} - \frac{p}{\pi}\left\{\tan^{-1}\frac{a/r\,.\,\sin(\theta+\alpha)}{1 - a/r\,.\,\cos(\theta+\alpha)} - \tan^{-1}\frac{a/r\,.\,\sin(\theta-\alpha)}{1 - a/r\,.\,\cos(\theta-\alpha)}\right\}$$
$$+\frac{p}{\pi}\left(1 - \frac{a^2}{r^2}\right)\frac{3-\sigma}{2}\sin\alpha\; a/r\,.\,\cos\,\theta$$
$$-\frac{p}{2\pi}\left(1 - \frac{a^2}{r^2}\right)\left\{\frac{a/r\,.\,\sin(\theta+\alpha)}{1 - 2a/r\,.\,\cos(\theta+\alpha) + a^2/r^2} - \frac{a/r\,.\,\sin(\theta-\alpha)}{1 - 2a/r\,.\,\cos(\theta-\alpha) + a^2/r^2}\right\}$$
$$\ldots \ldots \quad (10.31)$$

$$\widehat{\theta\theta} = \frac{p\alpha}{\pi}\frac{a^2}{r^2} - \frac{p}{\pi}\left\{\tan^{-1}\frac{a/r\,.\,\sin(\theta+\alpha)}{1 - a/r\,.\,\cos(\theta+\alpha)} - \tan^{-1}\frac{a/r\,.\,\sin(\theta-\alpha)}{1 - a/r\,.\,\cos(\theta-\alpha)}\right\}$$
$$+\frac{p}{2\pi}\left(1 - \frac{a^2}{r^2}\right)\left\{\frac{a/r\,.\,\sin(\theta+\alpha)}{1 - 2a/r\,.\,\cos(\theta+\alpha) + a^2/r^2} - \frac{a/r\,.\,\sin(\theta-\alpha)}{1 - 2a/r\,.\,\cos(\theta-\alpha)\, a^2/r^2}\right\}$$
$$+\frac{p}{\pi}\left(1 + \frac{a^2}{r^2}\right)\frac{3-\sigma}{2}\sin\alpha\,.\,\frac{a}{r}\cos\,\theta, \quad \ldots \ldots \quad (10.32)$$

$$\widehat{r\theta} = \frac{p}{2\pi}\left(1 - \frac{a^2}{r^2}\right)^2\left\{\frac{1}{1 - 2a/r\,.\,\cos(\theta+\alpha) + a^2/r^2} - \frac{1}{1 - 2a/r\,.\,\cos(\theta-\alpha) + a^2/r^2}\right\}$$
$$+\frac{p}{\pi}\left(1 - \frac{a^2}{r^2}\right)\frac{3-\sigma}{2}\sin\alpha\,.\,\frac{a}{r}\sin\,\theta. \quad \ldots \ldots \quad (10.33)$$

Values of the stresses have been calculated for $\alpha = 30°$, $\sigma = 0{\cdot}25$, and they are given in Table V; the corresponding graphs are drawn in fig. 6. A noticeable feature is the discontinuity of the boundary value of $\widehat{\theta\theta}$ due, of course, to the discontinuity of the applied traction. This discontinuity has been decidedly "smoothed out" at $r/a = 1{\cdot}2$.

TABLE V.—*Stresses for normal traction uniformly distributed over the arc* $-30° < \theta < 30°$.

θ.	0°.	20°.	30°.	40°.	60°.	90°.	120°.	150°.	180°.
r/a.					$\widehat{rr}$.				
1·0	—3·142	—3·142	—3·142						
			0	0	0	0	0	0	0
1·2	2·716	2·418	1·233	—0·048	+0·211	+0·192	+0·061	+0·000	—0·023
1·5	1·946	1·562	0·952	0·337	0·154	0·202	0·113	0·012	+0·001
2·0	1·185	0·926	0·641	0·345	0·043	0·184	0·133	0·065	0·037
3·0	0·593	0·491	0·380	0·258	—0·042	0·109	0·133	0·097	0·083
5·0	0·268	0·240	0·196	0·160	0·055	0·045	0·083	0·088	0·086
					$\widehat{\theta\theta}$.				
1·0	—0·719	—0·802	—0·905						
			+2·238	+2·100	+1·735	+1·047	+0·360	+0·144	—0·328
1·2	0·075	+0·196	0·442	0·676	0·935	0·752	0·281	0·120	0·272
1·5	+0·109	0·299	0·299	0·337	0·458	0·407	0·189	0·008	0·084
2·0	0·217	0·210	0·196	0·165	0·197	0·197	0·102	0·001	0·041
3·0	0·130	0·130	0·110	0·100	0·088	0·073	0·035	0·008	0·027
5·0	0·062	0·071	0·054	0·048	0·038	0·023	0·004	0·013	0·022
					$\widehat{r\theta}$.				
1·0	0	0	0	0	0	0	0	0	0
1·2	0	—0·573	—0·726	—0·552	—0·007	+0·157	+0·145	+0·085	0
1·5	0	0·977	1·163	1·008	0·205	0·129	0·173	0·110	0
2·0	0		0·617		0·263	0·051	0·138	0·097	0
3·0	0		0·279		0·210	—0·031	0·055	0·040	0
5·0	0		0·106		0·101	0·045	0·003	0·015	0

Comparing with the solution for a pressure on a segment of the boundary of a semi-infinite sheet, it is seen, however, that the curvature of the boundary introduces no essential modification in the way in which the stresses are "smoothed out." By making $\alpha \to 0$ in this example, in such a way that $pa\alpha \to \mathrm{F}$, we reproduce the results of the last paragraph.

(11) *Normal pressure* $\propto \cos\theta$, *acting over half the boundary.*

If a rivet just fills the hole, and is pulled sideways, *i.e.*, in the plane of the plate, then, neglecting the frictional forces on the boundary, and taking the rivet to be so hard that the compression it suffers is negligible, the resulting displacement in the plate is proportional to $\cos\theta$, over that half towards the direction of the pull. It will, therefore, be not far from true to take the normal force to be proportional to $\cos\theta$ over that half, and zero over the other half.

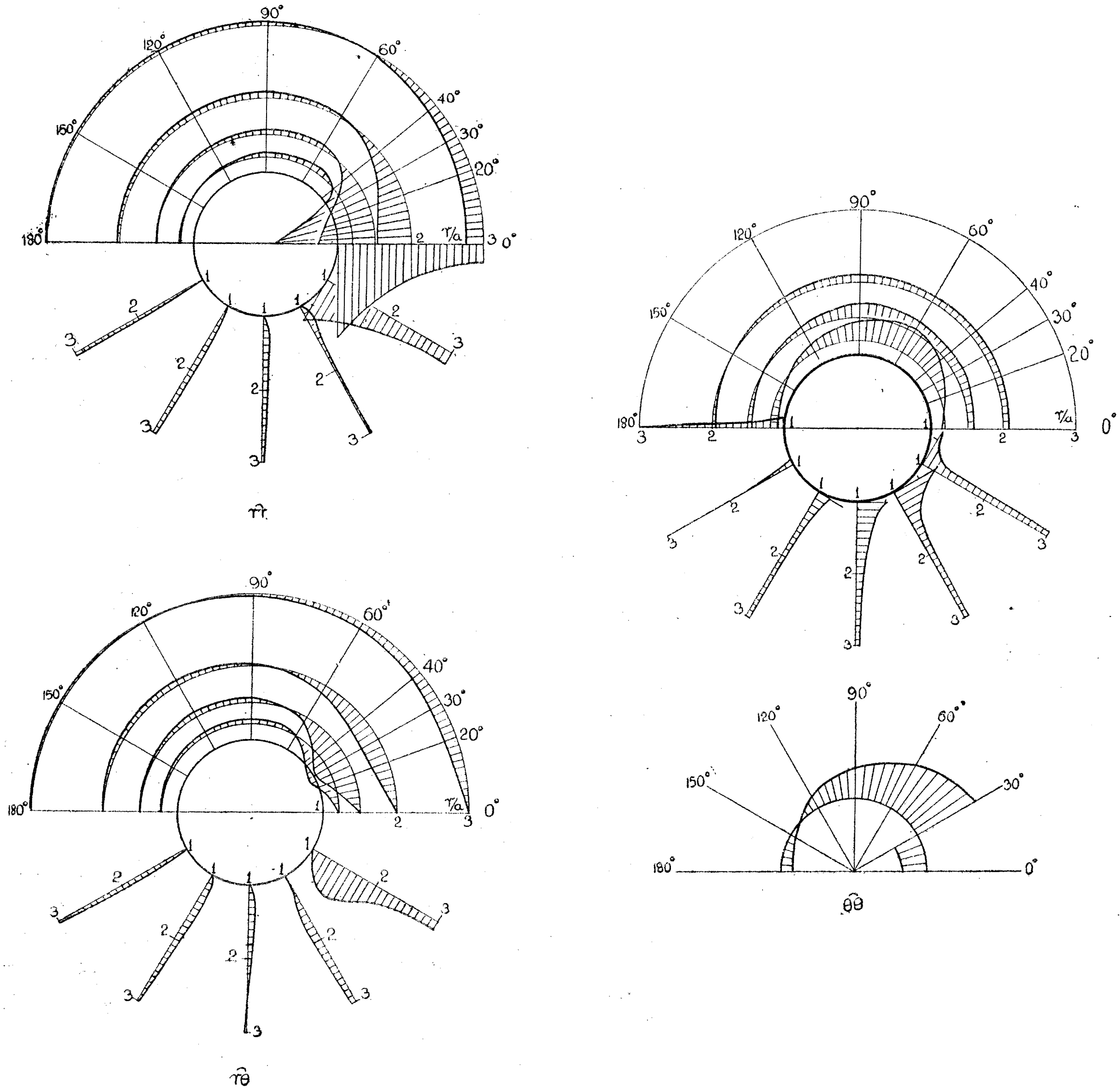

Fig. 6.

So, taking $\widehat{rr}_e = -p \,.\, \cos\theta$ when $|\theta| < \frac{\pi}{2}$ and $\widehat{rr}_e = 0$ when $|\theta| > \frac{\pi}{2}$, we obtain

$$\left.\begin{aligned} R_0 = \frac{p}{\pi}, \quad R_1 = \frac{p}{2}, \quad R_{2n} &= \frac{p}{\pi}\left\{\frac{\sin\frac{1}{2}(2n-1)\pi}{2n-1} + \frac{\sin\frac{1}{2}(2n+1)\pi}{2n+1}\right\} \\ R_{2n+1} &= 0. \end{aligned}\right\} \quad \ldots \quad (11.11)$$

The "singularities" are

$$P = \frac{p}{\pi}, \qquad X = \frac{\pi p a}{2}. \quad \ldots\ldots\ldots\ldots \quad (11.12)$$

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Consequently,

$$V = \frac{p}{2}\frac{a}{r}e^{-\iota\theta} + \frac{p}{\pi}\left(\frac{r}{a}e^{\iota\theta} + \frac{a}{r}e^{-\iota\theta}\right)\tan^{-1}\left(\frac{a}{r}e^{\iota\theta}\right), \quad . \quad . \quad . \quad . \quad . \quad (11.2)$$

$$\phi = \frac{p}{2}\frac{a}{r}\cos\theta + \frac{p}{2\pi}\left(\frac{r}{a}+\frac{a}{r}\right)\cos\theta \,.\, \tan^{-1}\frac{2a/r\,.\,\cos\theta}{1-a^2/r^2}$$
$$+\frac{p}{4\pi}\left(\frac{r}{a}-\frac{a}{r}\right)\sin\theta\log\frac{1+2a/r\,.\,\sin\theta+a^2/r^2}{1-2a/r\,.\,\sin\theta+a^2/r^2}, \quad . \quad . \quad . \quad . \quad . \quad (11.21)$$

$$\phi' = -\frac{p}{2}\frac{a}{r}\sin\theta + \frac{p}{2\pi}\left(\frac{r}{a}-\frac{a}{r}\right)\sin\theta\tan^{-1}\frac{2a/r\,.\,\cos\theta}{1-a^2/r^2}$$
$$-\frac{p}{4\pi}\left(\frac{r}{a}+\frac{a}{r}\right)\cos\theta\log\frac{1+2a/r\,.\,\sin\theta+a^2/r^2}{1-2a/r\,.\,\sin\theta+a^2/r^2}, \quad . \quad . \quad . \quad . \quad (11.22)$$

whence

$$\widehat{rr} = \frac{p}{2\pi}\left(1-\frac{a^2}{r^2}\right) - \frac{p}{2}\frac{a}{r}\cos\theta\left\{1-\frac{1-\sigma}{4}\left(1-\frac{a^2}{r^2}\right)\right\}$$
$$-\frac{p}{4\pi}\frac{r}{a}\cos\theta\,.\left(1+4\frac{a^2}{r^2}-\frac{a^4}{r^4}\right)\tan^{-1}\frac{2a/r\,.\,\cos\theta}{1-a^2/r^2}$$
$$-\frac{p}{2\pi}\frac{r}{a}\sin\theta\left(1-\frac{a^2}{r^2}\right)^2\log\frac{1+2a/r\,.\,\sin\theta+a^2/r^2}{1-2a/r\,.\,\sin\theta+a^2/r^2}, \quad . \quad . \quad . \quad (11.31)$$

$$\widehat{\theta\theta} = \frac{p}{2\pi}\left(3+\frac{a^2}{r^2}\right) + \frac{1-\sigma}{8}p\frac{a}{r}\cos\theta\,.\left(1+\frac{a^2}{r^2}\right)$$
$$-\frac{p}{4\pi}\frac{r}{a}\cos\theta\,.\left(3+\frac{a^4}{r^4}\right)\tan^{-1}\frac{2a/r\,.\,\cos\theta}{1-a^2/r^2}$$
$$-\frac{p}{2\pi}\frac{r}{a}\sin\theta\left(3-2\frac{a^2}{r^2}-\frac{a^4}{r^4}\right)\log\frac{1+2a/r\,.\,\sin\theta+a^2/r^2}{1-2a/r\,.\,\sin\theta+a^2/r^2}, \quad (11.32)$$

$$\widehat{r\theta} = \frac{1-\sigma}{8}\frac{a}{r}\sin\theta\,.\left(1-\frac{a^2}{r^2}\right)$$
$$-\frac{p}{4\pi}\frac{r}{a}\sin\theta\,.\left(1-\frac{a^4}{r^4}\right)\tan^{-1}\frac{2a/r\,.\,\cos\theta}{1-a^2/r^2}$$
$$+\frac{p}{2\pi}\frac{r}{a}\cos\theta\,.\left(1-\frac{a^2}{r^2}\right)^2\log\frac{1+2a/r\,.\,\sin\theta+a^2/r^2}{1-2a/r\,.\,\sin\theta+a^2/r^2}. \quad . \quad . \quad . \quad (11.33)$$

Values of the stresses, calculated from these formulæ, with $\sigma = 0\cdot25$, are given in Table VI, and the corresponding graphs in fig. 7. The graph of $\widehat{\theta\theta}$ has a sharp peak at $\theta = 90°$, attaining there a value about $0\cdot65$ of the maximum compression stress.

TABLE VI.—*Normal traction* $\propto$ cos θ, $|\theta| < 90°$.

θ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.
r/a.				$\widehat{rr}$.			
1·0	−2·000	−1·732	−1·000	0	0	0	0
1·2	1·535	1·302	0·674	+0·054	+0·122	+0·059	+0·037
1·5	1·141	0·957	0·461	0·058	0·135	0·077	0·052
2·0	0·762	0·629	0·289	0·040	0·140	0·115	0·098
3·0	0·435	0·357	0·167	0·026	0·111	0·124	0·121
4·0	0·296	0·242	0·118	0·013	0·088	0·113	0·116
5·0	0·222	0·184	0·091	0·008	0·073	0·100	0·106
				$\widehat{\theta\theta}$.			
1·0	+0·648	+0·739	+0·961	+1·273	+0·586	+0·083	−0·102
1·2	0·547	0·598	0·696	0·652	0·431	0·139	0·018
1·5	0·379	0·398	0·418	0·363	0·238	0·086	+0·019
2·0	0·247	0·258	0·233	0·182	0·116	0·055	0·017
3·0	0·134	0·129	0·108	0·077	0·039	0·008	−0·005
4·0	0·088	0·082	0·066	0·041	0·016	−0·004	0·012
5·0	0·064	0·059	0·045	0·025	0·006	0·008	0·014
				$\widehat{r\theta}$.			
1·0	0	0	0	0	0	0	0
1·2	0	−0·033	−0·045	+0·048	+0·128	+0·080	0
1·5	0	0·041	0·044	0·070	0·165	0·110	0
2·0	0	0·027	0·015	0·070	0·136	0·097	0
3·0	0	0·007	+0·010	0·056	0·086	0·072	0
4·0	0	0·000	0·015	0·044	0·061	0·044	0
5·0	0	+0·004	0·017	0·036	0·046	0·032	0

(12) *Normal pressure* $\propto \sqrt{1 - \theta^2/\alpha^2}$ *over the part of the boundary where* $|\theta| < \alpha$.

Our next example will be an approximation to the distribution round a rivet which is too small for the hole. In such a case, the pressure will be a maximum at the mid-point of the arc of contact, and will diminish to zero on both sides. By analogy with HERTZ's solution of the problem of the pressure between bodies in contact, we may take it that a formula which will represent the main features of the distribution is

$$\widehat{rr}_e = -p\sqrt{1 - \theta^2/\alpha^2} \qquad \text{when } |\theta| < \alpha$$
$$= 0 \qquad\qquad |\theta| > \alpha.$$

For this assumed law of pressure, the FOURIER coefficients are—

$$R_0 = \tfrac{1}{4}\alpha p, \qquad R_m = p/m \,.\, J_1(m\alpha) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12.11)$$

J_1 being the BESSEL function of the first kind and of the first order. The singularities are—

$$P = \tfrac{1}{4}\alpha p, \qquad X = \pi a p \, J_1(\alpha) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12.12)$$

The series for V has not been summed, so we give the formulæ for the stresses as infinite series. They are—

$$\widehat{rr} = -\tfrac{1}{2}p \sum_1^\infty J_1(m\alpha). \left\{\frac{2}{m} + \left(1 - \frac{a^2}{r^2}\right)\right\}\frac{a^m}{r^m}\cos m\theta$$
$$- \tfrac{1}{4}\alpha p \frac{a^2}{r^2} + \frac{3-\sigma}{4} p \, J_1(\alpha) \,.\left(1 - \frac{a^2}{r^2}\right)\frac{a}{r}\cos\theta \quad . \; . \; . \quad (12.21)$$

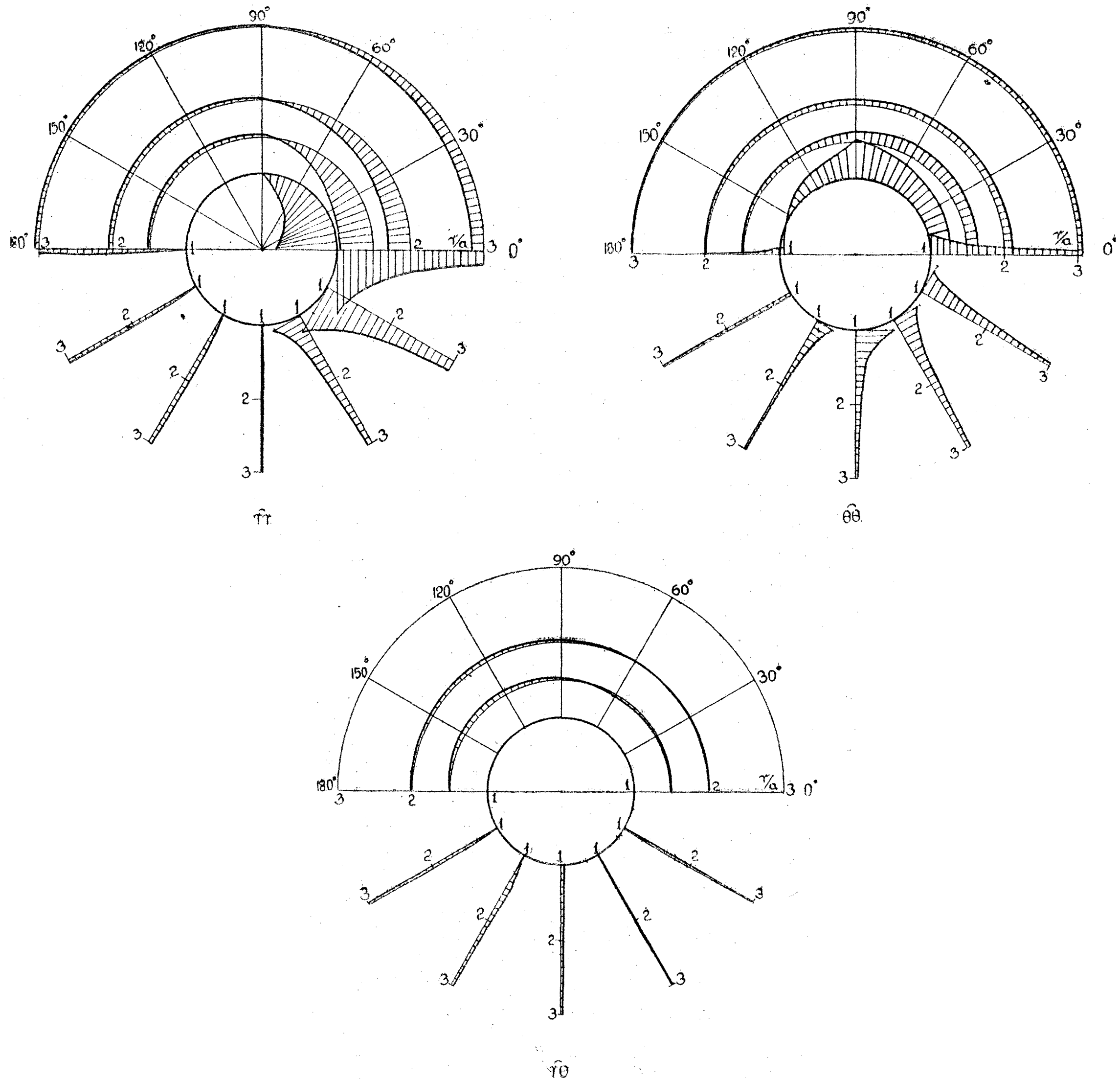

Fig. 7.

$$\widehat{\theta\theta} = -\tfrac{1}{2}p \sum_1^\infty \mathrm{J}_1(m\alpha) \, . \left\{\frac{2}{m} - \left(1 - \frac{a^2}{r^2}\right)\right\}\frac{a^m}{r^m}\cos m\theta$$

$$+ \tfrac{1}{4}\,\alpha p\left(1 + \frac{a^2}{r^2}\right) + \frac{3-\sigma}{4}\,p\,\mathrm{J}_1(\alpha) \, . \left(1 + \frac{a^2}{r^2}\right)\frac{a}{r}\cos\theta \qquad (12.22)$$

$$\widehat{r\theta} = -\tfrac{1}{2}\,p \sum_1^\infty \mathrm{J}_1(m\alpha) \, . \left(1 - \frac{a^2}{r^2}\right)\frac{a^m}{r^m}\sin m\theta$$

$$+ \frac{3-\sigma}{4}\,p\,\mathrm{J}_1(\alpha) \, . \left(1 - \frac{a^2}{r^2}\right)\frac{a}{r}\sin\theta. \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (12.23)$$

Values of these are given in Table VII with $\alpha = 1$ radian, and the corresponding graphs in fig. 8. Again, the hoop-stress, $\widehat{\theta\theta}$, has a peak at the end of the arc of contact, its value there being nearly as great as the maximum radial pressure.

TABLE VII.—*Normal pressure* $\propto \sqrt{1 - \frac{\theta^2}{\alpha^2}}$, $|\theta| < \alpha$. $\alpha = 1^c = 57 \cdot 3°$.

θ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.
r/a.				$\widehat{rr}$.			
1·0	−2·272	−1·935	0	0	0	0	0
1·5	1·365	1·065	−0·129	+0·183	+0·128	+0·048	+0·021
2·0	0·903	0·676	0·135	0·144	0·144	0·091	0·066
3·0	0·493	0·372	0·108	0·075	0·119	0·113	0·095
4·0	0·329	0·254	0·090	0·044	0·098	0·103	0·101
5·0	0·242	0·190	0·074	0·029	0·080	0·094	0·095
				$\widehat{\theta\theta}$.			
1·0	+0·239	+0·392	+1·824	+1·136	+0·449	−0·055	−0·239
1·5	0·280	0·345	0·404	0·390	0·212	0·040	0·043
2·0	0·216	0·220	0·213	0·184	0·108	0·019	0·019
3·0	0·126	0·120	0·088	0·071	0·046	0·001	0·017
4·0	0·082	0·076	0·058	0·036	0·012	0·011	0·021
5·0	0·060	0·056	0·042	0·024	0·006	0·012	0·018
				$\widehat{r\theta}$.			
1·0	0	0	0	0	0	0	0
1·5	0	−0·140	−0·115	+0·155	+0·188	+0·116	0
2·0	0	0·095	0·050	0·122	0·170	0·110	0
3·0	0	0·039	0·010	0·070	0·104	0·071	0
4·0	0	0·016	+0·004	0·050	0·071	0·049	0
5·0	0	0·007	0·009	0·039	0·052	0·037	0

(13) *Tangential traction* $\propto \sin^3 \theta \cos \theta$.

So far, we have considered the effects of normal traction alone, but, in the case of a rivet, it seems more than probable that there will be some tangential friction between the rivet and the hole. Moreover, the analysis of the experimental results to be given leaves no doubt of the existence of frictional effects. The value of this friction can be seen to increase with θ on account of obliquity, but to depend also upon the normal pressure which decreases as θ increases. Consequently the friction will increase from zero at $\theta = 0$ to a maximum (in the neighbourhood of 60°), and then fall away to zero at 90°. From 90° to 180° it should be zero. The simplest formula to represent this is

$$\widehat{r\theta}_e = p \sin^3 \theta \cos \theta, \quad |\theta| < \tfrac{1}{2}\pi,$$
$$= 0, \quad |\theta| > \tfrac{1}{2}\pi.$$

giving

$$\left.\begin{aligned} U_m &= -\frac{4 \sin \frac{1}{2} m\pi}{\pi} \frac{m^2 - 10}{(m^2 - 4)(m^2 - 16)} \quad (m \text{ odd}), \\ U_2 &= \tfrac{1}{4}, \quad U_4 = \tfrac{1}{8}, \\ U_m &= 0, \quad (m \text{ even}, > 4) \end{aligned}\right\} \quad \ldots \ldots \quad (13.11)$$

$$X = 0 \cdot 4\, ap. \quad \ldots \ldots \ldots \ldots \quad (13.12)$$

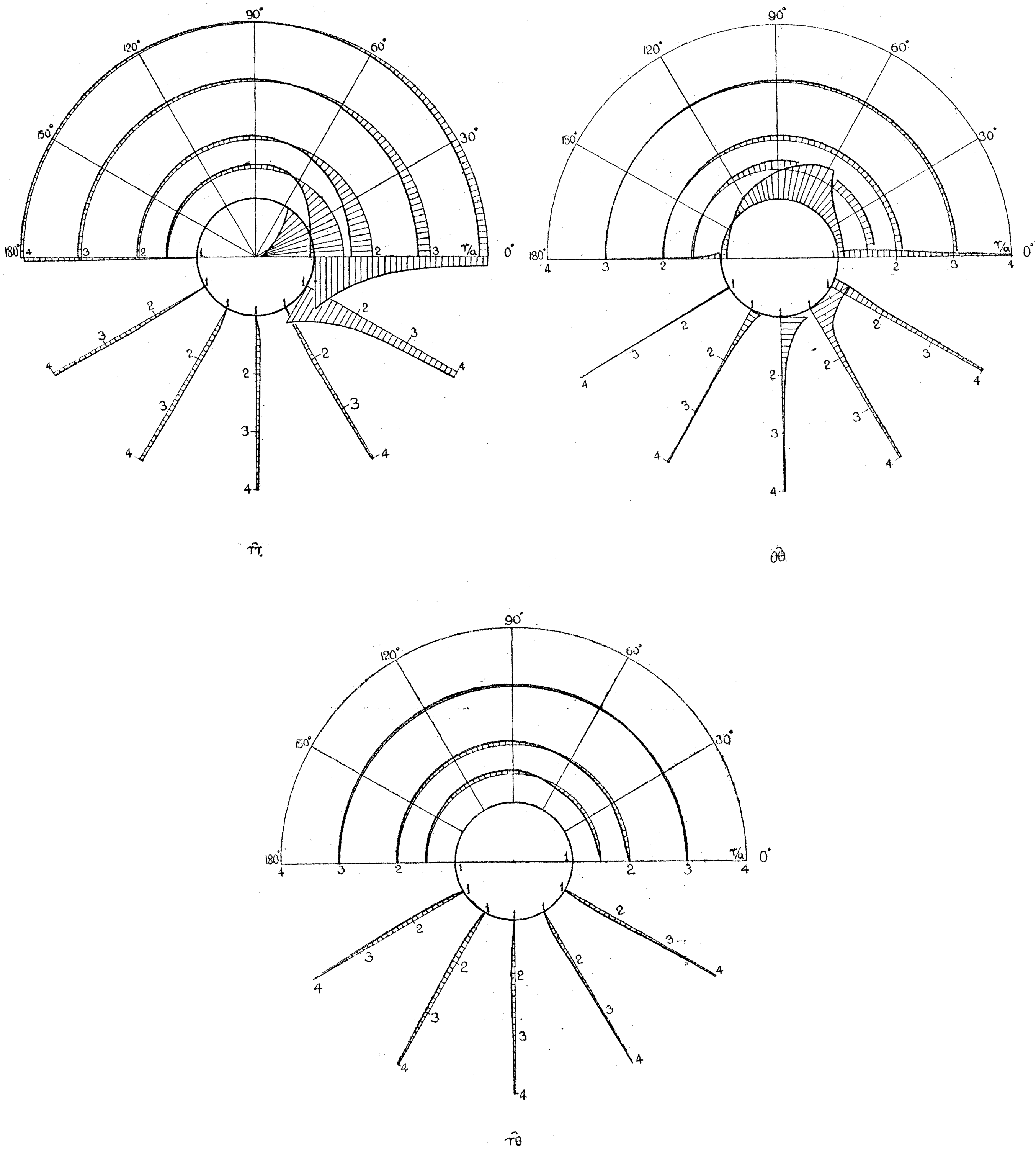

Fig. 8.

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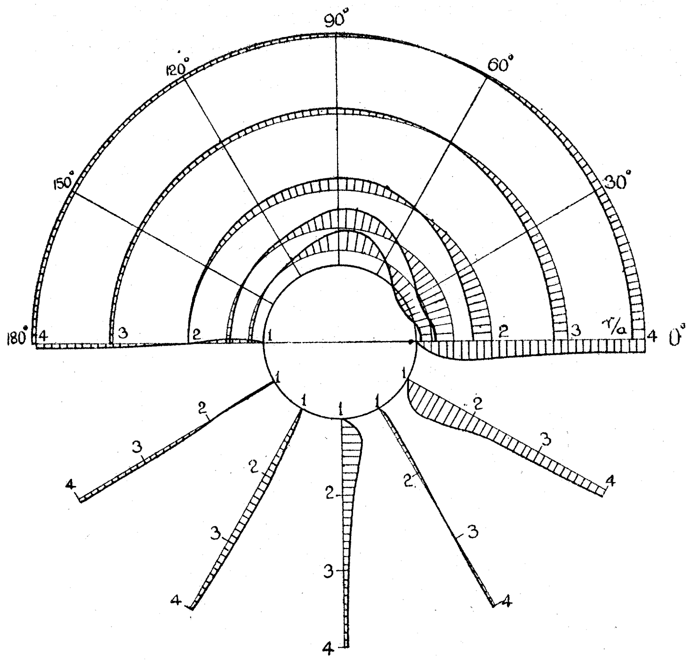
The series for W can be summed, but the result is too complicated to be of service in the calculations. Moreover, the stresses come out as relatively small differences, so that to obtain a final result at all accurate involves carrying the calculations to a large number of significant figures. Consequently the series were used in the calculations. The results are embodied in Table VIII, and the corresponding graphs are given in fig. 9.

TABLE VIII.—*Tangential traction* $\propto \sin^3 \theta \cos \theta$.

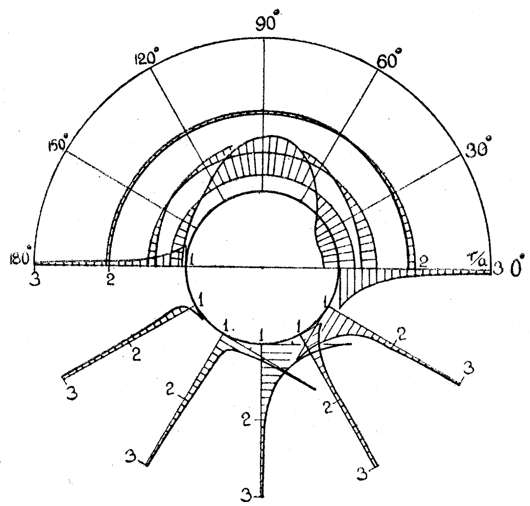
θ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.
				$\widehat{rr}$.			
r/a.							
1·0	0	0	0	0	0	0	0
1·2	—0·334	—0·608	+0·098	+0·634	+0·101	—0·026	—0·066
1·5	0·596	0·657	0·105	0·646	0·218	+0·010	0·051
2·0	0·608	0·508	0·045	0·437	0·254	0·071	+0·009
3·0	0·425	0·318	—0·017	0·210	0·195	0·108	0·070
4·0	0·304	0·226	0·035	0·120	0·144	0·106	0·085
5·0	0·231	0·175	0·038	0·078	0·112	0·095	0·085
				$\widehat{\theta\theta}$.			
1·0	—1·273	—2·079	+0·638	+2·945	+3·503	+0·591	—0·690
1·2	0·978	0·980	0·348	1·203	0·342	—0·222	0·403
1·5	0·492	0·315	0·151	0·420	0·204	0·105	0·212
2·0	0·147	0·058	0·061	0·115	0·070	0·057	0·114
3·0	+0·004	+0·019	0·028	0·020	0·000	0·040	0·060
4·0	0·025	0·027	0·021	0·006	—0·021	0·033	0·044
5·0	0·027	0·026	0·015	0·002	0·013	0·028	0·035
				$\widehat{r\theta}$.			
1·0	0	+0·850	+2·551	0	0	0	0
1·2	0	0·561	0·775	+0·657	+0·282	+0·126	+0
1·5	0	0·208	0·168	0·448	0·364	0·180	0
2·0	0	0·020	0·060	0·234	0·239	0·166	0
3·0	0	—0·023	0·002	0·103	0·162	0·106	0
4·0	0	0·016	—0·002	0·063	0·100	0·071	0
5·0	0	0·010	0·005	0·047	0·070	0·051	0

(14) *Composite distribution.*

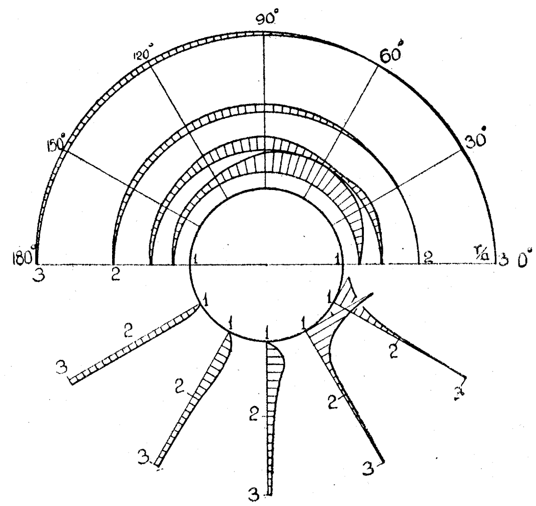
The final example will be an attempt to find the main features of the distribution of stress round a rivet, when pulled sideways. As has already been said, the distribution of the normal component of the boundary traction will be very nearly that considered in paragraph 11, and the frictional component will be sufficiently well represented by the formula of the preceding paragraph. Table IX has been calculated by adding 0·8 times the values of Table VI to 0·2 times the corresponding values of Table VIII. Graphs exhibiting the stresses are plotted in fig. 10. It will be seen that the stresses



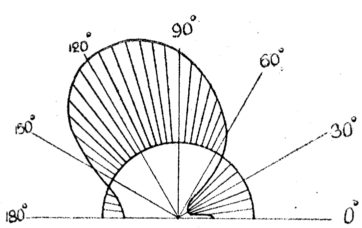
$\widehat{rr}$



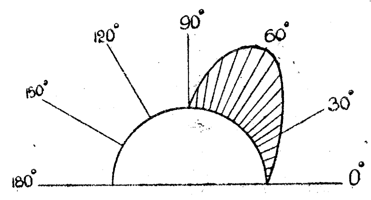
$\widehat{\theta\theta}$



$\widehat{r\theta}$



Edge values.



Edge values.

Fig. 9.

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on the circle $r/a = 1 \cdot 2$ in fig. 10 have a very great resemblance to those of the experimentally determined stresses given in Part II, as have also those along the radii $\theta = 0°$ and $\theta = 90°$.

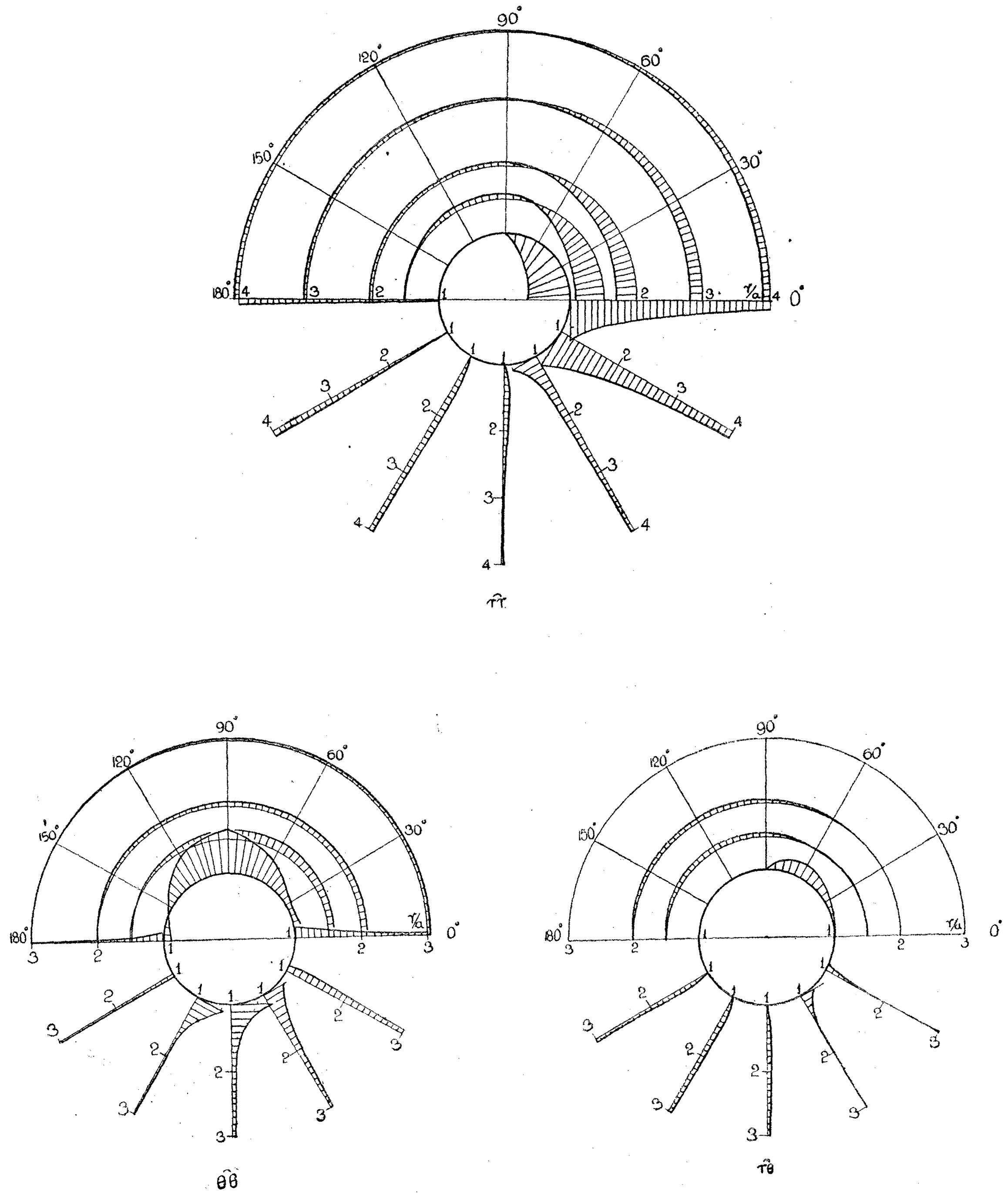

Fig. 10.

TABLE IX.—*Composite distribution.*

θ.	0°.	30°.	60°.	90°.	120°.	150°.	180°.
r/a.				$\widehat{rr}$.			
1·0	−1·600	−1·385	−0·800	0	0	0	0
1·2	1·295	1·163	0·520	+0·170	+0·118	+0·043	+0·016
1·5	1·032	0·897	0·348	0·176	0·152	0·064	0·031
2·0	0·731	0·605	0·222	0·119	0·163	0·106	0·080
3·0	0·433	0·349	0·137	0·063	0·128	0·121	0·111
4·0	0·298	0·239	0·101	0·034	0·099	0·112	0·110
5·0	0·224	0·182	0·080	0·022	0·081	0·099	0·102
				$\widehat{\theta\theta}$.			
1·0	+0·264	+0·170	+0·896	+1·607	+0·169	+0·185	−0·220
1·2	0·242	0·282	0·626	0·762	0·413	0·067	0·095
1·5	0·205	0·256	0·365	0·374	0·231	0·048	0·029
2·0	0·168	0·195	0·199	0·169	0·107	0·033	0·009
3·0	0·108	0·106	0·093	0·066	0·031	−0·002	0·016
4·0	0·075	0·071	0·057	0·034	0·009	0·010	0·018
5·0	0·057	0·052	0·030	0·020	0·002	0·012	0·018
				$\widehat{r\theta}$.			
1·0	0	+0·170	+0·510	0	0	0	0
1·2	0	0·086	0·119	+0·179	+0·159	+0·089	0
1·5	0	0·009	−0·002	0·146	0·205	0·124	0
2·0	0	−0·018	0·000	0·103	0·157	0·111	0
3·0	0	0·010	+0·008	0·065	0·101	0·079	0
4·0	0	0·003	0·012	0·048	0·069	0·049	0
5·0	0	+0·001	0·012	0·038	0·051	0·036	0

PART II.

Comparison with experimental results.

Although it is impossible to carry out an experiment on an infinite plate, it is, nevertheless, of interest to compare the foregoing analysis with results of experiments, and some results obtained by COKER and FUKUDA are available. These were mentioned with some details of the experimental method in a lecture delivered by Dr. COKER on the occasion of the centenary of the Franklin Institute, and this lecture is reprinted in the 'Journal of the Franklin Institute,' vol. 199, March, 1925, and graphs are there given of the stress distributions. Load was applied to a steel pin 0·312 inches in diameter, surrounded by a xylonite ring turned to fit a hole 0·766 inches in diameter in a plate of xylonite 18 inches by 6 inches, the plate being supported by a row of pins near its upper (short) edge. Stresses were measured (by combining optical and

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extensometer measurements) near the boundary of the hole, and along the vertical and horizontal through the centre of the hole. We propose to see how far the analysis of Part I enables us to reproduce the stresses in the horizontal and vertical from those near the boundary of the hole.

The measurements near the hole (at 0·01 inches from the boundary) lead to the following table (Table X), p and q being the principal stresses in lbs./sq. inch. (The angles are measured from the line of load, to conform with Part I, a convention which differs by 90° from that actually employed by Dr. Coker.)

Table X.

$\theta°$.	0°.	10°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.
$p+q$	−1380	−1320	−1140	−1120	− 950	− 695	− 208	+ 465	1250
$p-q$	+1770	1710	1650	1570	1570	1810	2070	2030	1870

$\theta°$.	90°.	100°.	110°.	120°.	130°.	140°.	150°.	160°.	170°.	180°.
$p+q$	1580	1600	1270	880	548	314	157	−51	−240	−318
$p-q$	1570	1250	995	736	517	378	199	0	−119	−199

The first step was the calculation of the Fourier coefficients for $(p+q)$ and $(p-q)$ from these values. It was found that after the sixth they were small (within the probable limits of experimental error), and in any case the higher harmonics will be considerably affected by experimental error, and can have little, if any, significance. The first six only were therefore taken. These are :—

$$p+q = 78 - 700\cos\theta - 1088\cos 2\theta + 195\cos 3\theta + 297\cos 4\theta + \cos 5\theta - 139\cos 6\theta,$$

$$p-q = 1144 + 832\cos\theta - 429\cos 2\theta - 95\cos 3\theta + 49\cos 4\theta + 145\cos 5\theta + 37\cos 6\theta.$$

Now, while $p+q = \widehat{rr} + \widehat{\theta\theta}$, $p-q$ does *not* equal $\widehat{rr} - \widehat{\theta\theta}$, unless the principal axes of stress are radial and transversal. As, however, for the purpose in hand when the experiments were performed it was not necessary, the inclinations of these principal axes were not measured, a long and laborious process of successive approximation had to be resorted to to determine $\widehat{rr} - \widehat{\theta\theta}$ and $\widehat{r\theta}$. From formulæ (5.2), (5.3) and (5.4), restricting ourselves to the cosine terms in $\widehat{rr}$ and $\widehat{\theta\theta}$, and to the sine terms in $\widehat{r\theta}$, it will be seen that if we write—

$$\widehat{rr} = \Sigma\, \widehat{rr}_n \cos n\theta, \qquad \widehat{\theta\theta} = \Sigma\, \widehat{\theta\theta}_n \cos n\theta, \qquad \widehat{r\theta} = \Sigma\, \widehat{r\theta}_n \sin n\theta,$$

then

$$2\widehat{r\theta}_n = \widehat{rr}_n - \widehat{\theta\theta}_n \qquad (n \neq 1)$$

$$2\widehat{r\theta}_1 = (\widehat{rr}_1 - \widehat{\theta\theta}_1) - \frac{3-\sigma}{1+\sigma}(rr_1 + \theta\theta_1).$$

Taking $p - q$ as an approximation to $\widehat{rr} - \widehat{\theta\theta}$, an approximate value of the $\widehat{r\theta}$ coefficients can be obtained, and an approximation to $\widehat{r\theta}$ calculated from them. This value of $\widehat{r\theta}$ can be used to obtain a better value of $\widehat{rr} - \widehat{\theta\theta}$, using the formula $\widehat{rr} - \widehat{\theta\theta} = \{(p-q)^2 - (2\widehat{r\theta})^2\}^{\frac{1}{2}}$. This is then analysed and the result is a second approximation to the $\widehat{rr} - \widehat{\theta\theta}$ coefficients. This leads to a better approximation to the $\widehat{r\theta}$ coefficients—and the process is continued until the constancy of the $\widehat{r\theta}$ coefficients indicates convergence to their true values. The ultimate result obtained is—

$$\widehat{rr} = -505\cdot5 - 787\cdot5\cos\theta - 348\cos2\theta + 107\cos3\theta + 100\cos4\theta$$
$$- 71\cdot5\cos5\theta - 76\cos6\theta.$$

$$\widehat{\theta\theta} = 583\cdot5 + 87\cdot5\cos\theta - 740\cos2\theta + 88\cos3\theta + 197\cos4\theta$$
$$+ 72\cdot5\cos5\theta - 63\cos6\theta.$$

$$\widehat{r\theta} = 227\cdot5\sin\theta + 196\sin2\theta + 9\cdot5\sin3\theta - 485\sin4\theta - 72\sin5\theta$$
$$- 6\cdot5\sin6\theta.$$

In the course of the numerical work, and in the result, two discrepancies occur. The first is that the value of $\widehat{r\theta}$ obtained does not vanish with $p - q$, as it should. The cause of this is difficult to trace. The value is greater than the probable experimental error, but considering that each Fourier coefficient involves all the experimental numbers, that the higher harmonics, though small, may not be entirely negligible, and the nature of the process of approximation, this discrepancy is not regarded as being sufficiently serious to discredit the results entirely. The other discrepancy is that the numerical value of $p + q$ contains a constant term 78, while the formula (5.54) contains *no* constant term. The analysis in Part I has proceeded upon the assumption that the plate is free from stress at infinity, but the finite plate of the experiments introduces edge effects. These cannot at present be taken fully into account, but the effect of limited width is important, and consideration of it enables us to account for this constant term. The plate, being limited laterally, will have zero transversal stress at infinity, in the direction $\theta = 0$, and to secure this we add terms to χ

$$\chi_1 = Ar^2 + Ba^2\log(r/a) + (Dr^2 + Ea^2 + Fa^4/r^2)\cos2\theta,$$

and choose the coefficients so that the boundary stresses are zero, and $\widehat{\theta\theta} = 0$ at infinity. The resulting values of the stresses are—

$$\widehat{rr} = 2A\{(1 - a^2/r^2) + (1 - 4a^2/r^2 + 3a^4/r^4)\cos 2\theta\}$$

$$\widehat{\theta\theta} = 2A\{(1 + a^2/r^2) - (1 + 3a^4/r^4)\cos 2\theta\}$$

$$\widehat{r\theta} = -2A(1 + 2a^2/r^2 - 3a^4/r^4)\sin 2\theta$$

$$\widehat{rr} + \widehat{\theta\theta} = 4A - 8Aa^2/r^2 . \cos 2\theta,$$

which is the same stress distribution as that obtained by KIRSCH for the effect of a hole in an infinite tension member. Comparing with the values obtained by our FOURIER analysis, we obtain—

$$4A = 78 \quad \text{or} \quad A = 19 \cdot 5.$$

Putting in the experimental values of r and a, *i.e.*, $0 \cdot 393''$ and $0 \cdot 383''$, we find that these additional terms account for—

$$\widehat{rr} = 2 - 3 \cdot 5 \cos 2\theta$$
$$\widehat{\theta\theta} = 76 - 144 \cdot 5 \cos 2\theta$$
$$\widehat{r\theta} = -7 \cdot 5 \sin 2\theta$$

leaving, as the values of the stresses round a circle of radius $0 \cdot 393''$ to be accounted for by the formulæ of Part I,

$$\widehat{rr} = -507 \cdot 5 - 787 \cdot 5 \cos\theta - 344 \cdot 5 \cos 2\theta + 107 \cos 3\theta + 109 \cos 4\theta - 71 \cdot 5 \cos 5\theta - 76 \cos 6\theta.$$

$$\widehat{\theta\theta} = 507 \cdot 5 + 87 \cdot 5 \cos\theta - 595 \cdot 5 \cos 2\theta + 88 \cos 3\theta + 197 \cos 4\theta + 72 \cdot 5 \cos 5\theta - 63 \cos 6\theta.$$

$$\widehat{r\theta} = 227 \cdot 5 \sin\theta + 203 \cdot 5 \sin 2\theta + 9 \cdot 5 \sin 3\theta - 48 \cdot 5 \sin 4\theta - 72 \sin 5\theta - 6 \cdot 5 \sin 6\theta.$$

To pass to the values of R_m and U_m we have, as is easily deduced from (5.51) and (5.53),

$$R_0 = -\widehat{rr}_0\, a^2/r^2$$

$$R_1 - U_1 = -r/a\,(\widehat{rr}_1 - \widehat{r\theta}_1), \quad R_1 = -\frac{r^3}{a^3}\{\widehat{rr}_1 + \tfrac{1}{4}(3 + \sigma)(\widehat{rr}_1 - \widehat{r\theta}_1)(1 + a^2/r^2)\}$$

$$R_m - U_m = -r^m a^{-m}(\widehat{rr}_n - \widehat{r\theta}_n), \quad R_m = -r^{m+2} a^{-m-2}\{\widehat{rr}_m + \tfrac{1}{2}(m+2)(\widehat{rr}_m - \widehat{r\theta}_m)(1 - a^2/r^2)\}$$

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where we must have $r/a = 0{\cdot}393/0{\cdot}383$, and $\sigma = 0{\cdot}357$, giving,

m.	R_m.	U_m.	$R_m - U_m$.
0	534	—	534
1	804	− 237	1041
2	321	− 256	577
3	− 105	− 3	− 102
4	− 91	74	− 165
5	86	86	0
6	76	− 5	81

As a partial check we can calculate the force resultant from the value of $R_1 - U_1$,

$$
\begin{aligned}
X &= \pi a\,(R_1 - U_1) \\
&= \pi \times 0{\cdot}383 \times 1041 \text{ lb./sq. inch of thickness,} \\
&= \pi \times 0{\cdot}383 \times 1041 \times 0{\cdot}157 \text{ lb. on thickness } 0{\cdot}157'', \\
&= 197 \text{ lb., nearly,}
\end{aligned}
$$

agreeing very well with the 200 lb. actually used in the experiments.

The above values lead to the following table of edge stresses.

TABLE XI.

θ°.	0°.	10°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.
$\widehat{rr}$	−1622	−1558	−1445	−1286	−1226	−1183	−1038	− 795	− 384
$\widehat{r\theta}$	0	20	93	234	405	538	567	480	267
$\widehat{\theta\theta}$	50	52	58	84	193	454	885	1394	1798

θ°.	90°.	100°.	110°.	120°.	130°.	140°.	150°.	160°.	170°.	180°.
$\widehat{rr}$	−46	135	111	134	− 21	− 48	− 42	− 71	− 50	− 58
$\widehat{r\theta}$	148	37	− 8	− 8	− 13	− 38	− 77	− 95	− 66	0
$\widehat{\theta\theta}$	1926	1738	1343	917	579	325	84	− 178	−406	−498

If the xylonite ring just fitted the hole, the values of $\widehat{rr}$ and $\widehat{r\theta}$ from 90° to 180° should be zero. Postive values of $\widehat{rr}$ at the edge are impossible, and the apparent values of $\widehat{r\theta}$ in the neighbourhood of 160° are an effect of the discrepancy already mentioned. All things considered, however, much better could hardly have been expected, and all these discrepancies are comparable in magnitude with the probable experimental error. In fig. 11, these stresses are plotted, along with the experimentally determined p and q; this figure shows the possible effect of inaccuracy in the distance from the

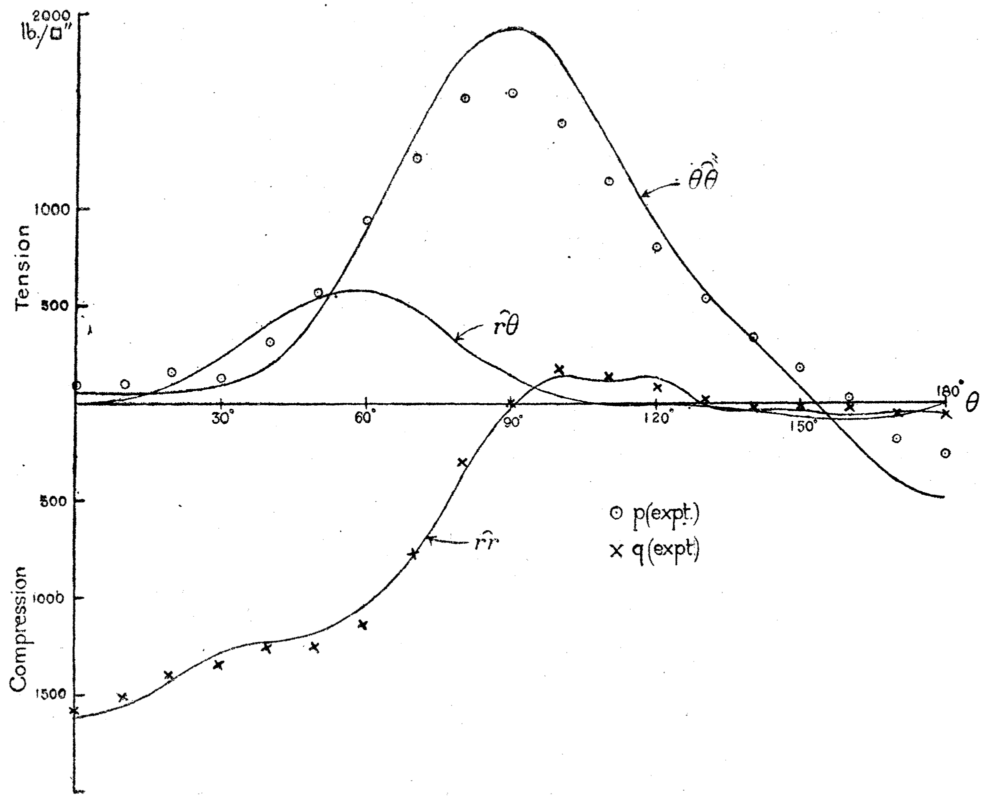


Fig. 11.

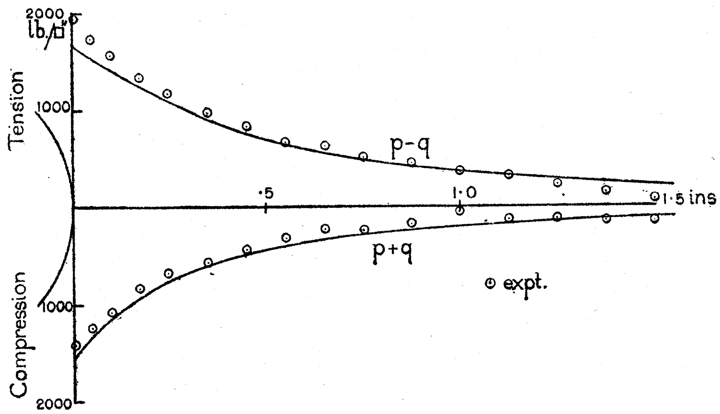
edge at which the stresses were measured, and of taking $\widehat{rr}$ and $\widehat{\theta\theta}$ as principal stresses, and that these effects may be considerable.

Values are also given in Table XII of the stresses calculated from the above, on the centre line below the hole, and across the section in line with the hole.

TABLE XII.

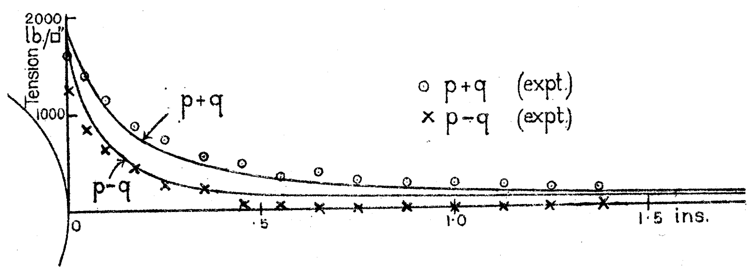
r/a.	Distance from edge (inches).	Centre line below hole.		Across section in line with hole.			
		$\widehat{rr}$.	$\widehat{\theta\theta}$.	$\widehat{rr}$.	$\widehat{\theta\theta}$.	$\widehat{r\theta}$.	$p-q$.
1·0	0·000	—1622	50	— 46	1926	— 148	2014
1·2	0·077	—1342	139	+ 208	993	— 161	848
1·5	0·191	—1039	170	221	518	— 142	412
2·0	0·383	— 716	152	154	275	— 73	190
3·0	0·766	— 397	100	77	152	— 51	126
4·0	1·149	— 250	70	45	117	— 39	107
5·0	1·532	— 169	53	29	102	— 31	96
6·0	1·915	— 118	41	20	95	— 26	91

Values of the sum and difference of the principal stresses are shown in figs. 12 and 13, along with the experimentally determined stresses, the continuous curves being the calculated ones. The general agreement is all that could be hoped for; calculated



Centre line below hole.
Fig. 12.

stresses derived from those on the boundary fall off with distance very much as the observed ones do. For the larger values of r there is a definite, though not great, divergence. The finiteness of the experimental plate, and the consequent edge effects, together with the modification of the stress distribution due to the method of supporting



Across section in line with hole.
Fig. 13.

the plate (designed to equalise the stresses at the top edge), are sufficient causes to account for this divergence; and, despite the additional terms introduced to remove the constant term in $p + q$, it cannot be claimed that edge effects are included in the calculations. Moreover, the divergence is comparable with the probable experimental error.

In conclusion, the author wishes to thank Dr. Coker for his continued interest in the work, despite several long delays, for permission to use the experimental results, and for valuable comments and suggestions which he has made from time to time; and also Messrs. A. G. G. de Chastelain, B.Sc., and H. H. Taylour for their assistance in preparing the figures.

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